Representative Families and Kernels

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Parameterized Complexity Summer School
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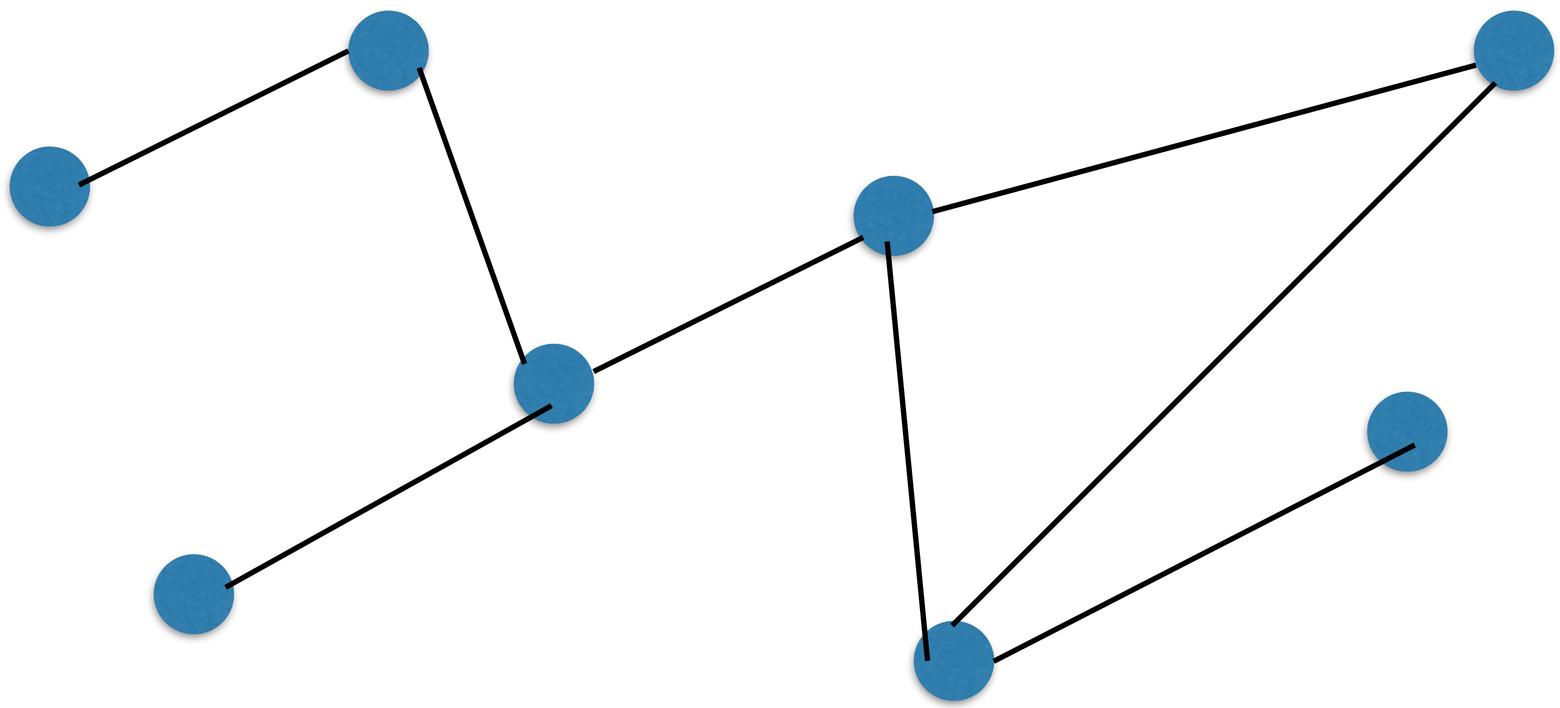
Outline

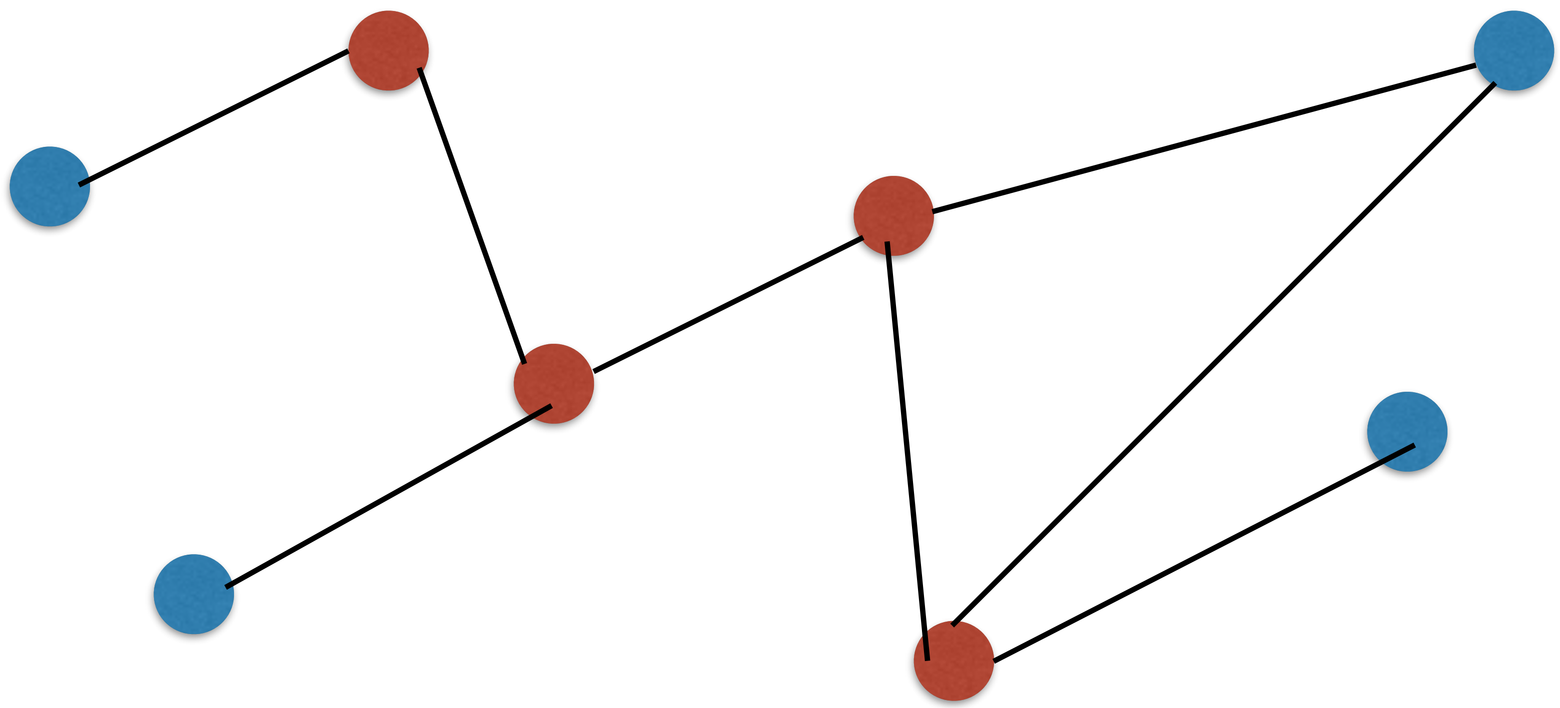
1. Vertex Cover, Representative Family, more applications, and its generalisation to matrices

2. Overview of an alternate randomised polynomial kernel for Vertex Cover above Maximum Matching

Vertex Cover



Vertex Cover



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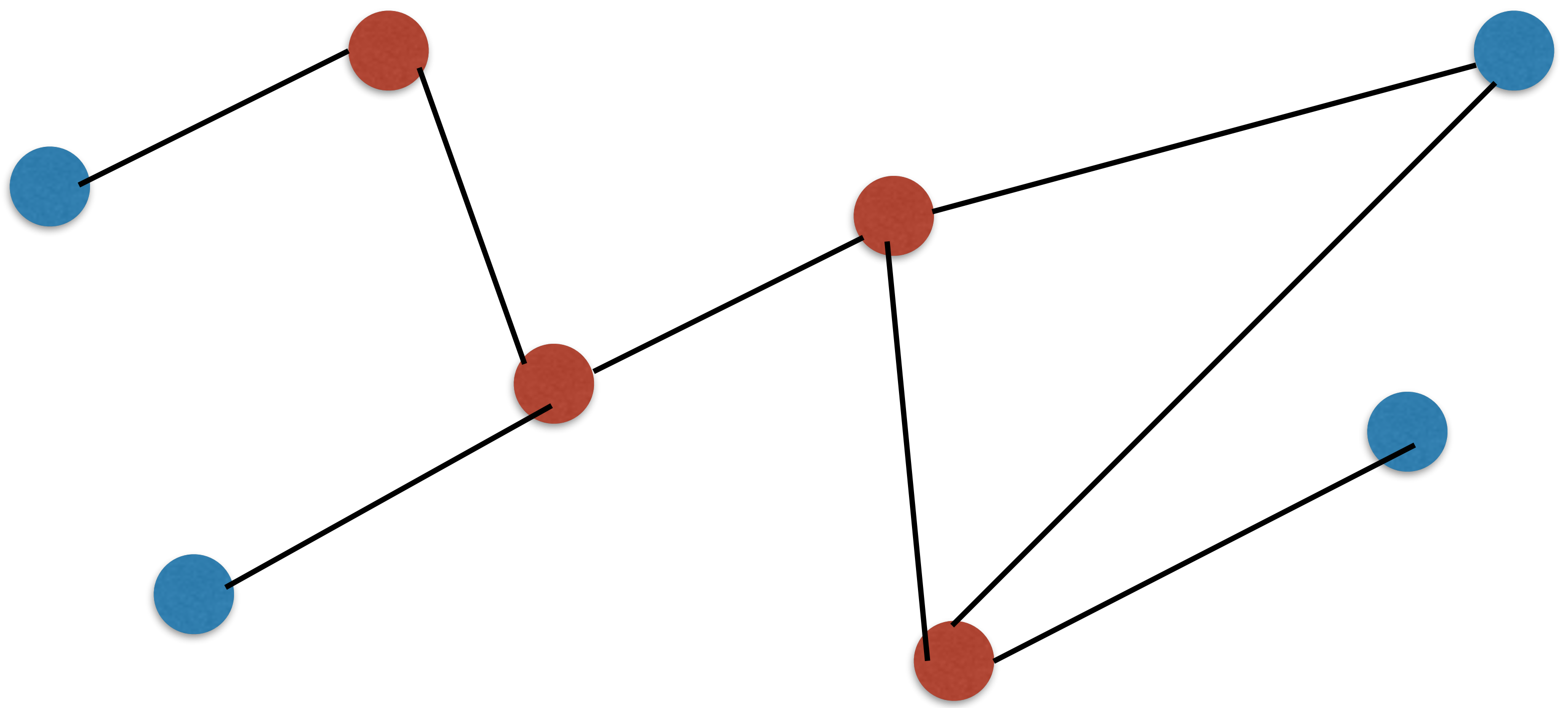
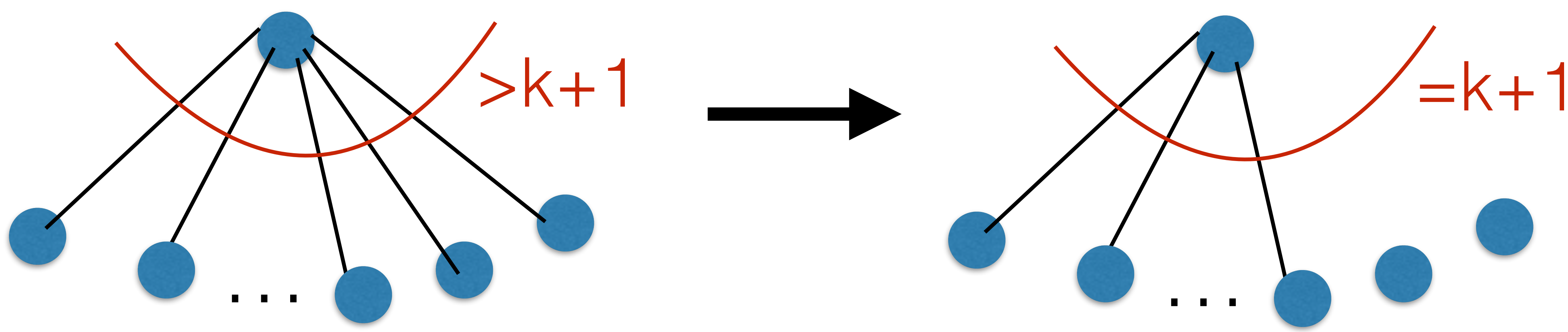


Input: A graph G and an integer k

Question : Is there a vertex cover of size k

A Simple Kernel for VC



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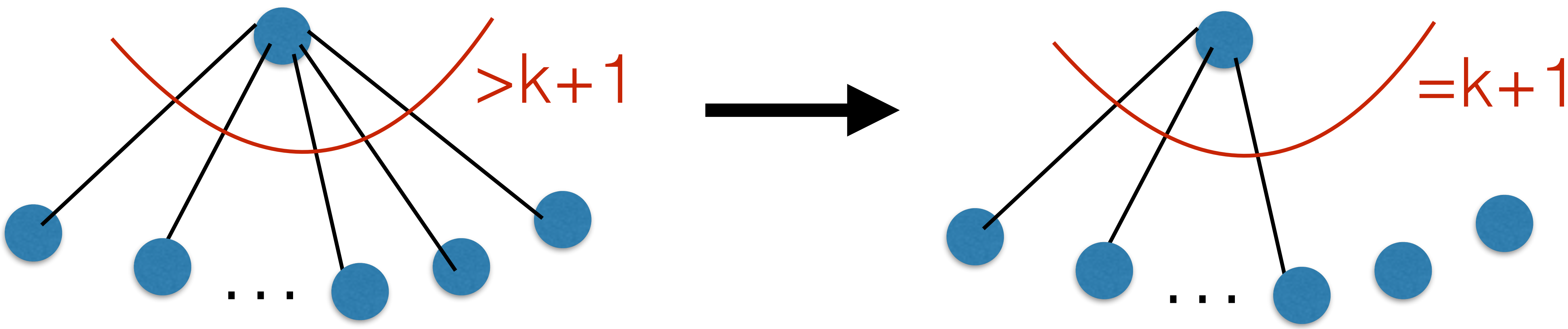

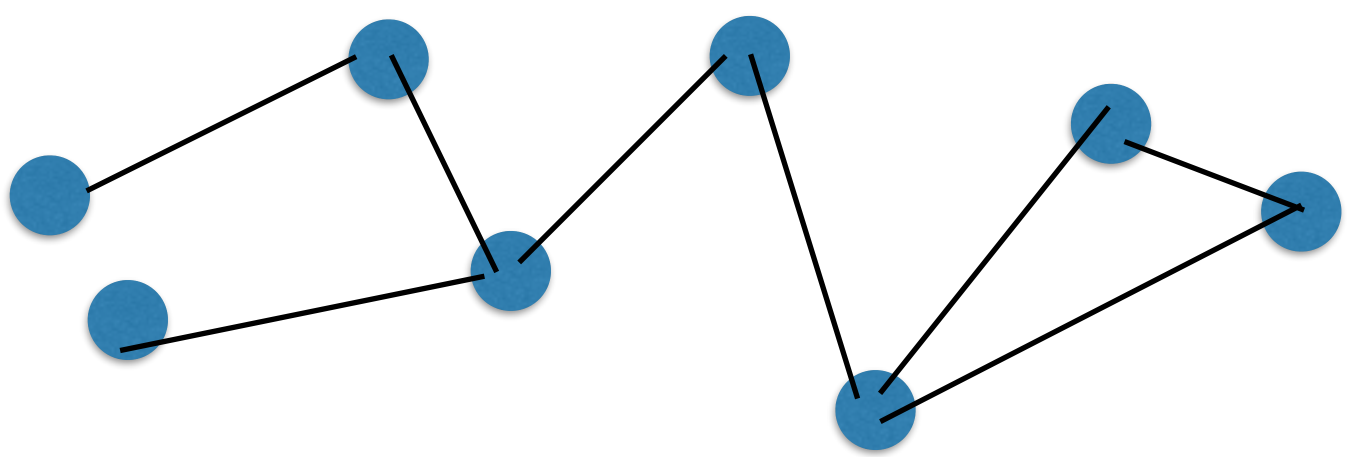

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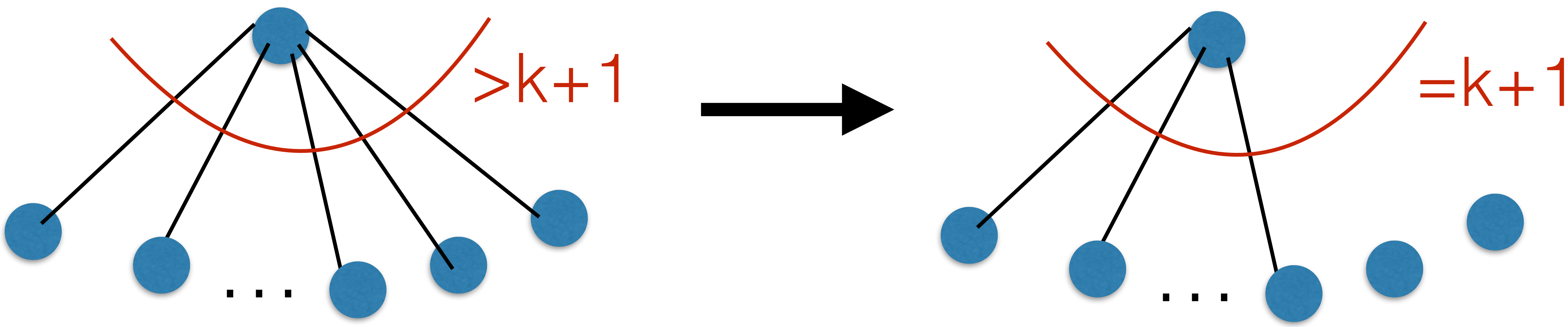
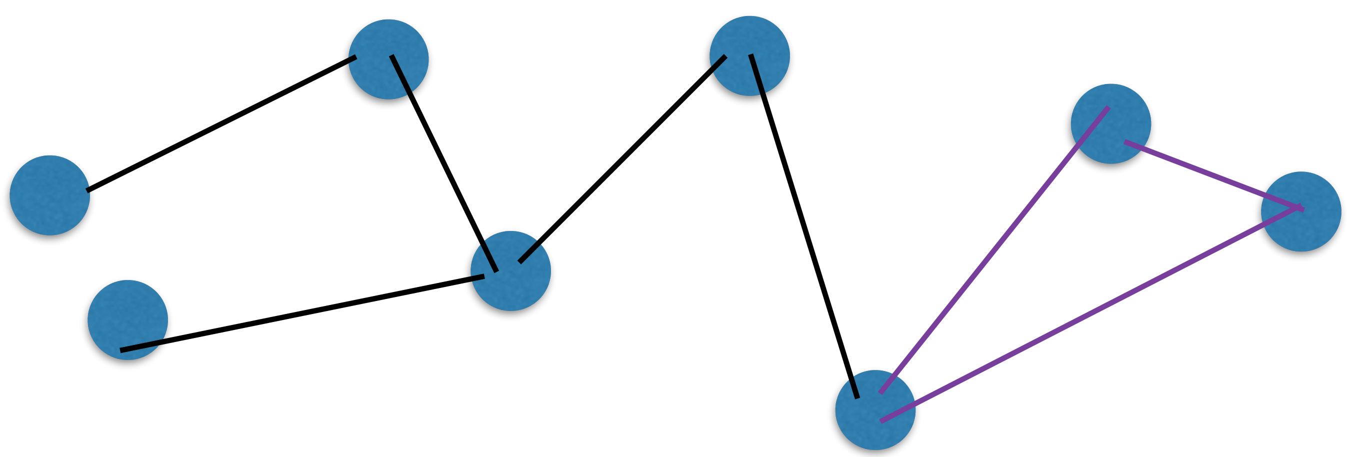
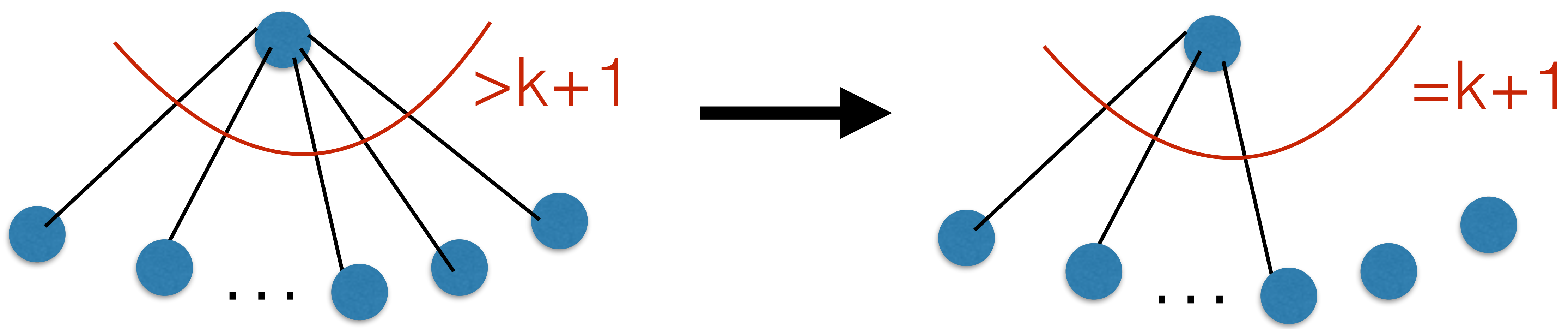
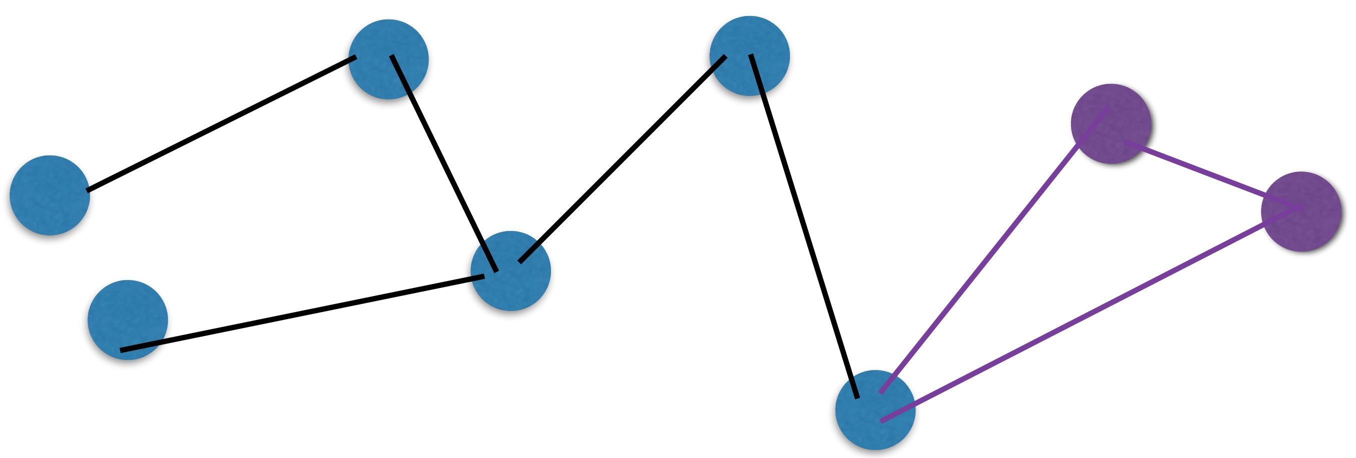




- Delete all but k(k+1)+1 edges

A Simple Kernel for VC





- Delete all but k(k+1)+1 edges
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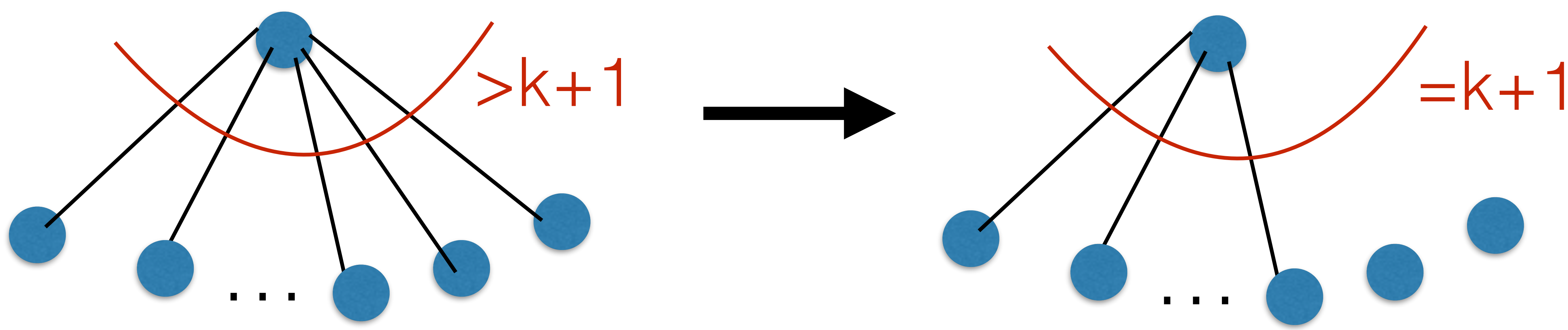

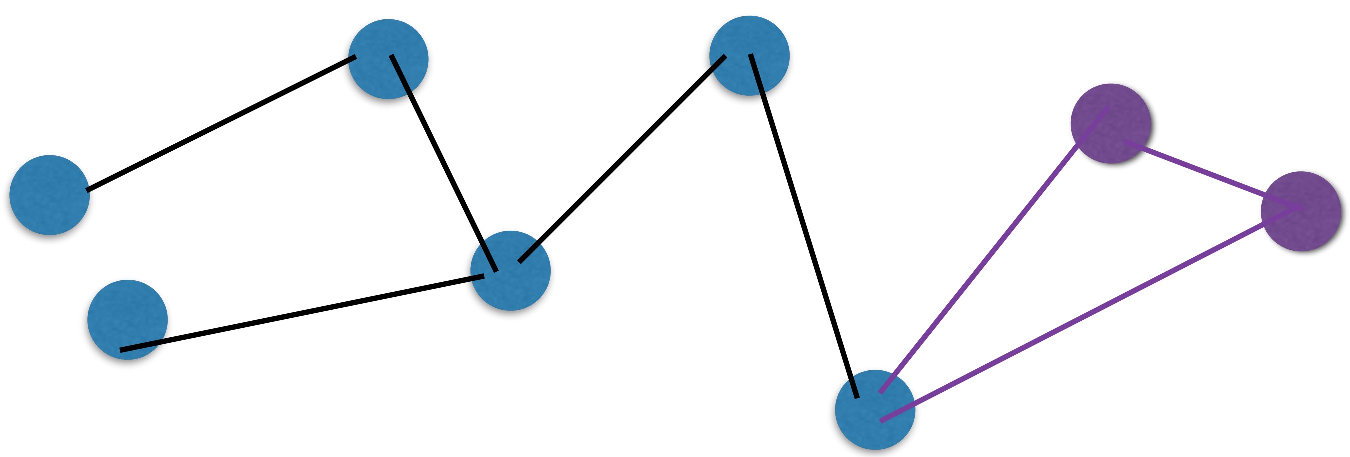

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- The resulting graph has size $O(k^2)$.

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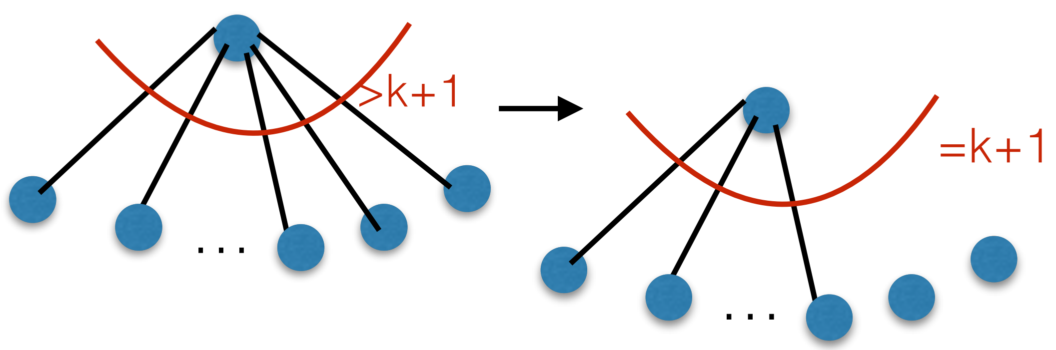

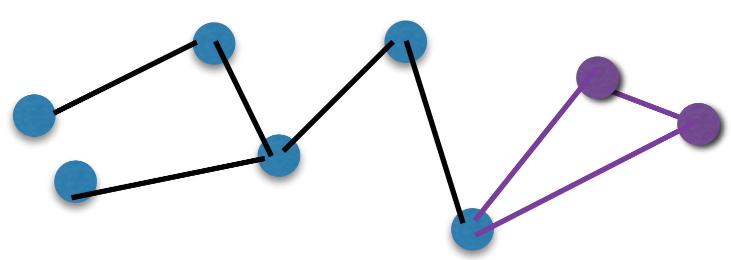

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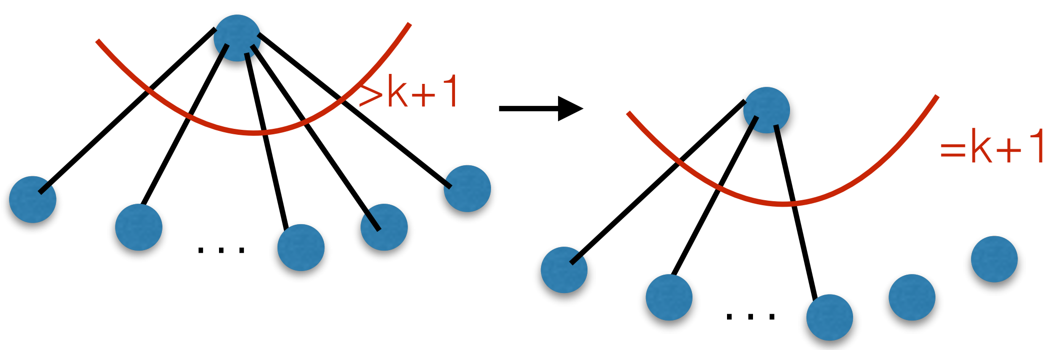
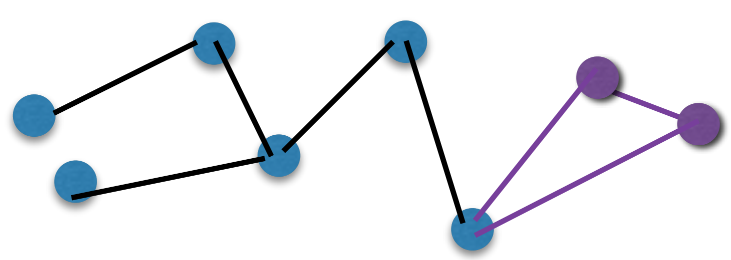
G ⟶ H

A Simple Kernel for VC

$(H,k) \equiv (G,k)$





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$G \longrightarrow H$

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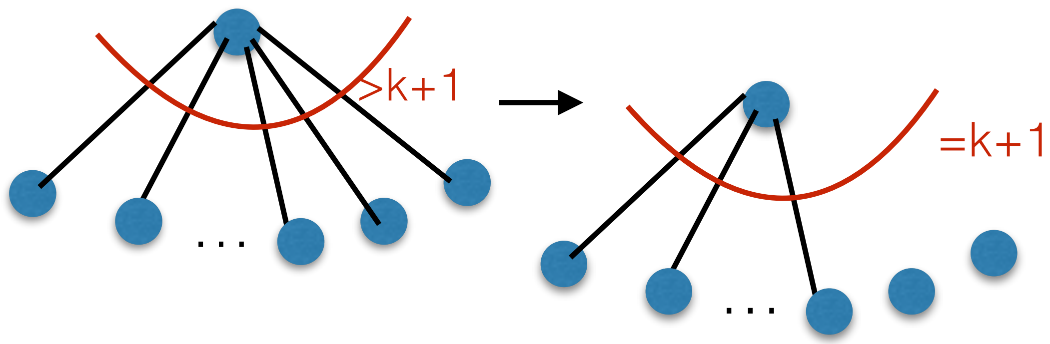

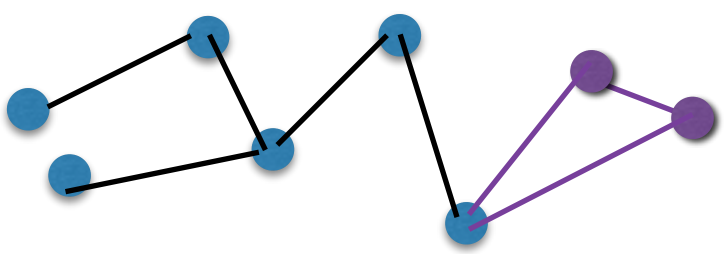

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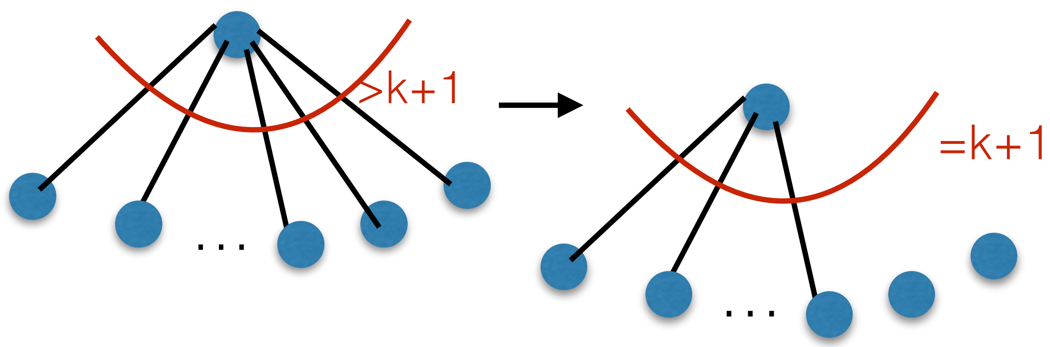
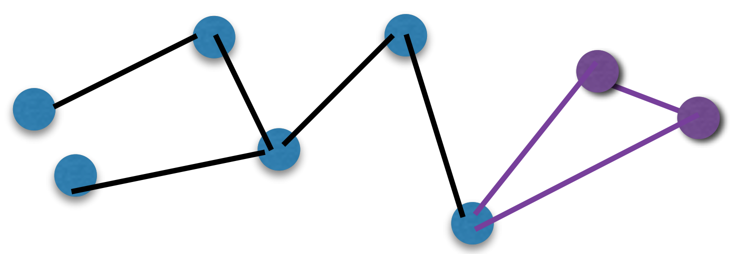
$(H,k) \equiv (G,k)$

$(\Rightarrow)$

If X is a V.C of G, then X is a V.C of H

A Simple Kernel for VC





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G $\longrightarrow$ H

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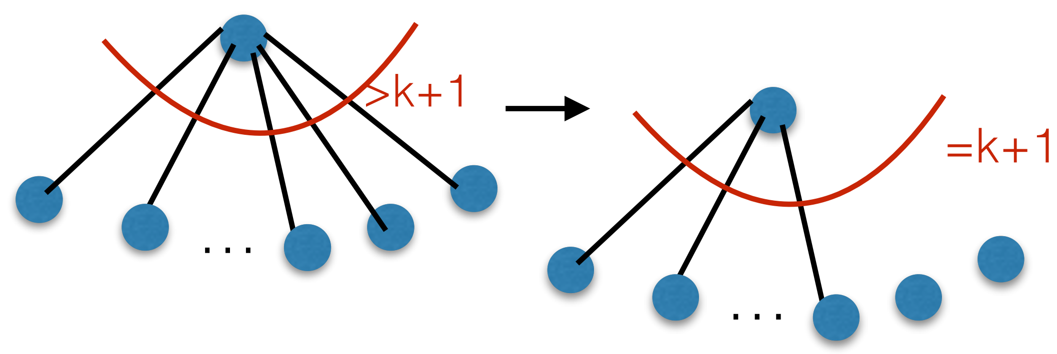
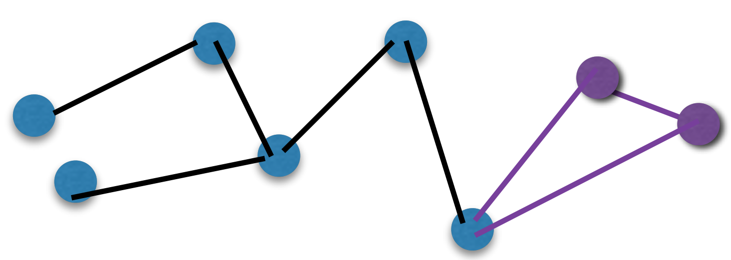
If X is a V.C of G, then X is a V.C of H

Proof: H is a subgraph of G

A Simple Kernel for VC





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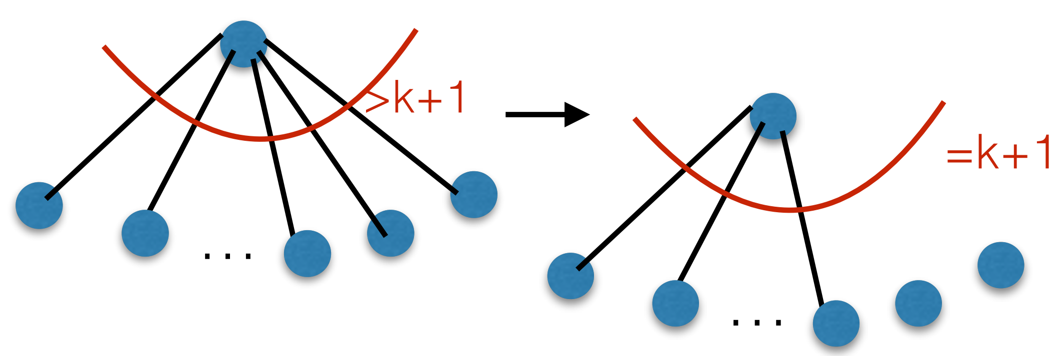
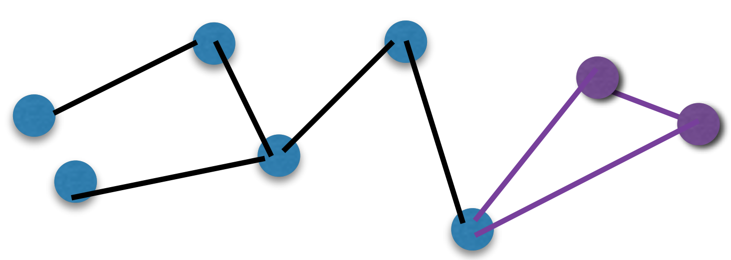
($\Leftarrow$)

Let X be a k size subset of V(H).
If X is a V.C of H, then X is a V.C of G

A Simple Kernel for VC





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Let X be a k size subset of $V(H)$.
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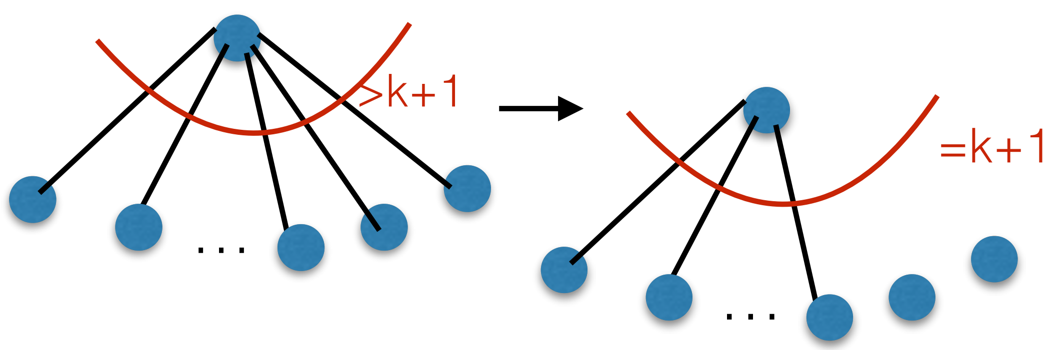
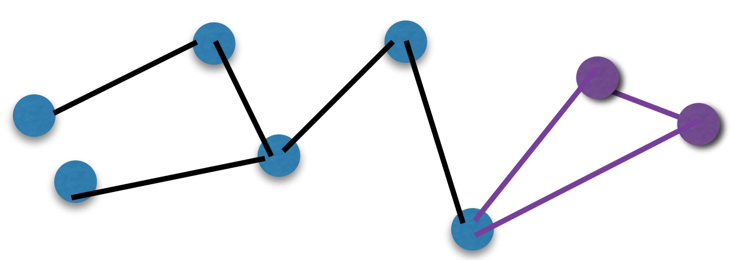
$\Updownarrow$

If X is not a V.C of G, then X is not a V.C of H

A Simple Kernel for VC





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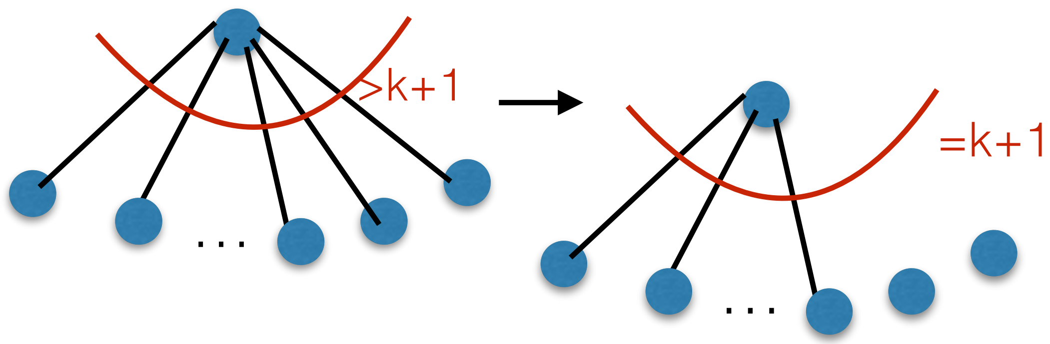

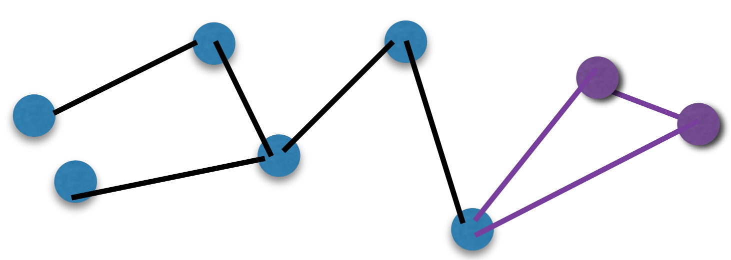

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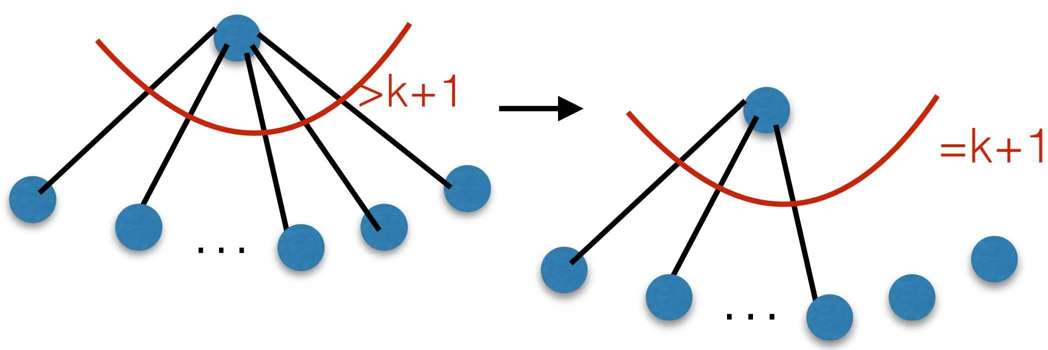
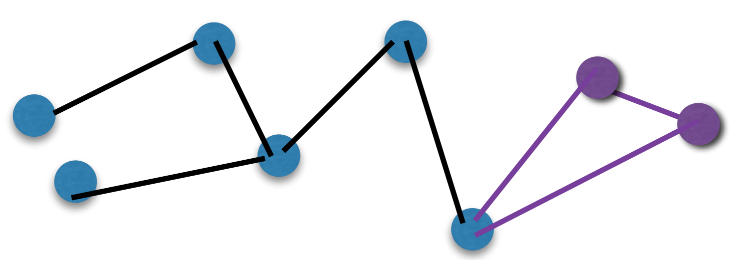
$$G \longrightarrow H$$

Let X be a k size subset of $V(H)$. If X is not a V.C of G, then X is not a V.C of H

$$G = G_0 \rightarrow G_1 \rightarrow G_2 \;\; . \; . \; . \; . \; G_t \rightarrow G_{t+1} = H$$

A Simple Kernel for VC





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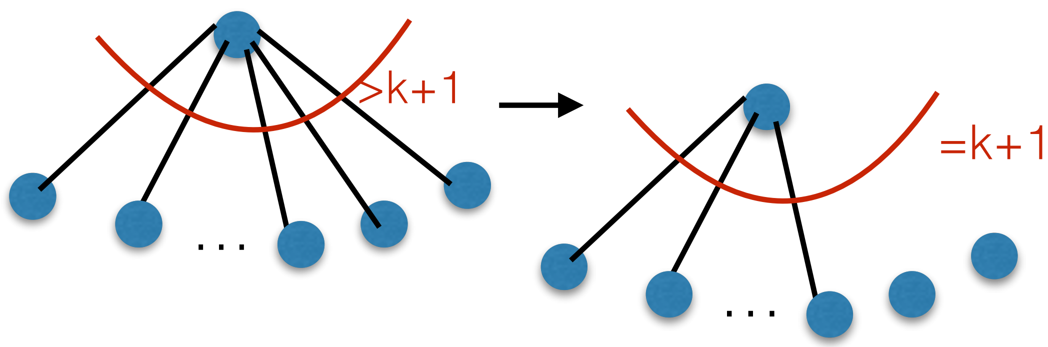

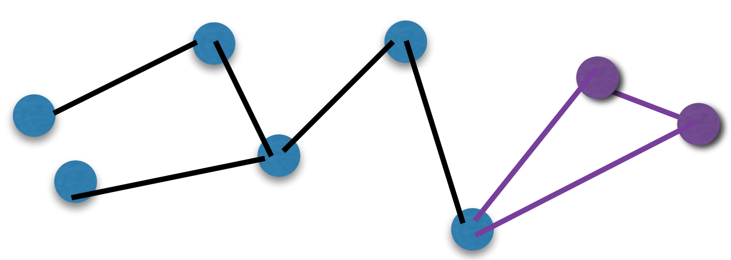

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Proof:

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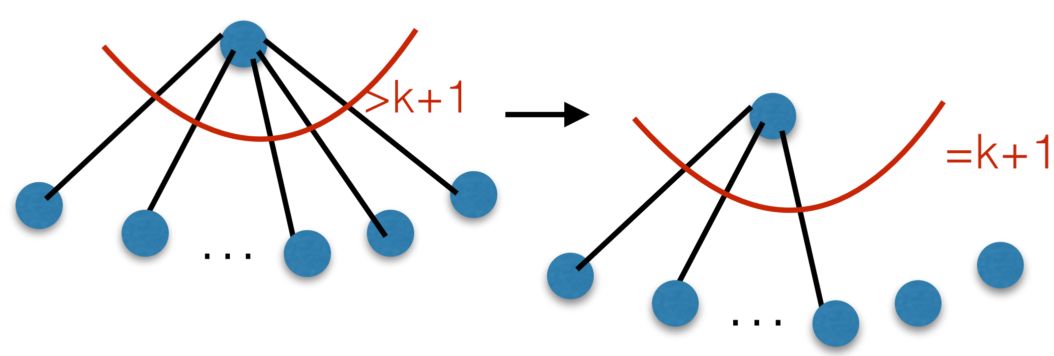

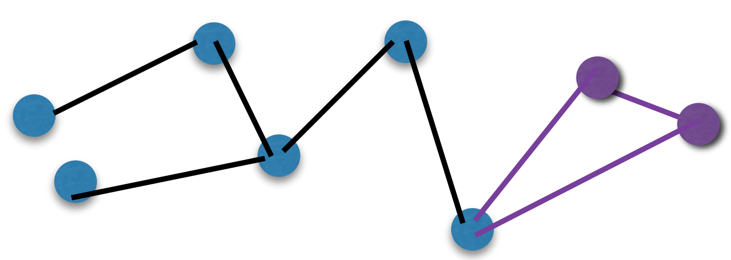

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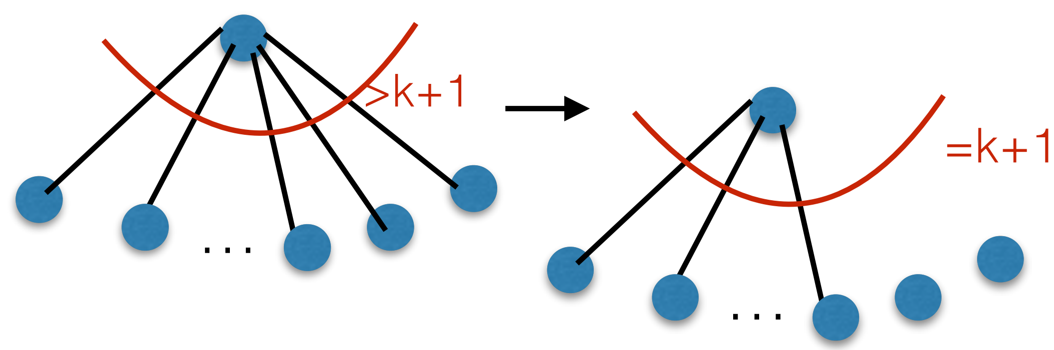

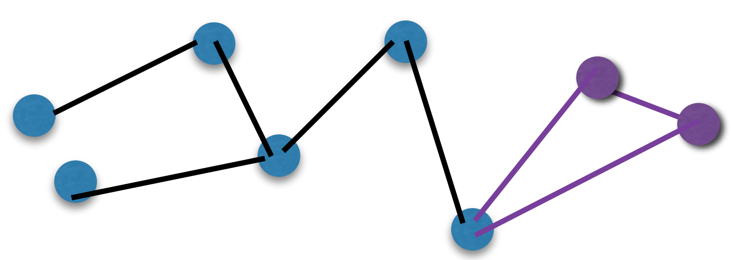

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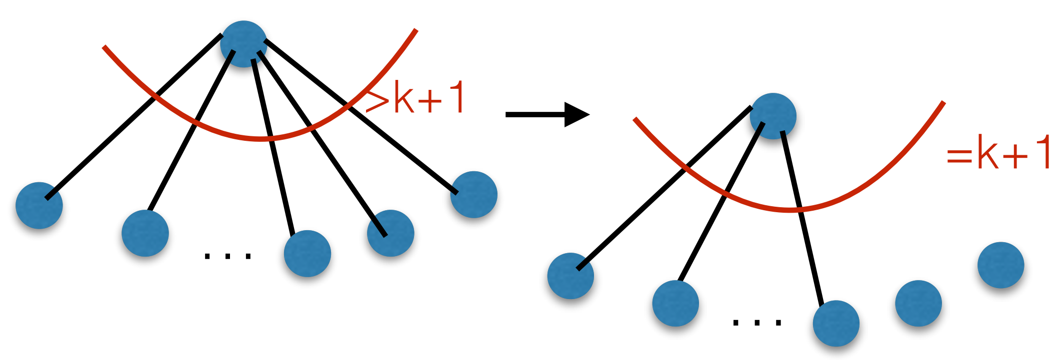

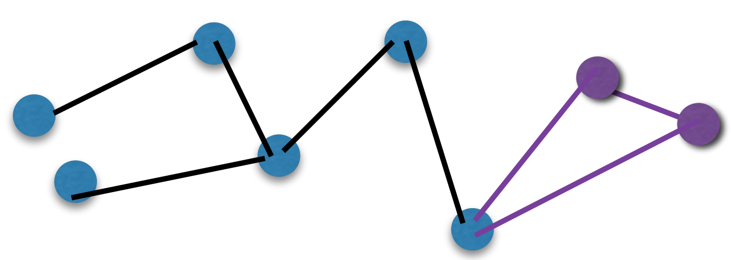

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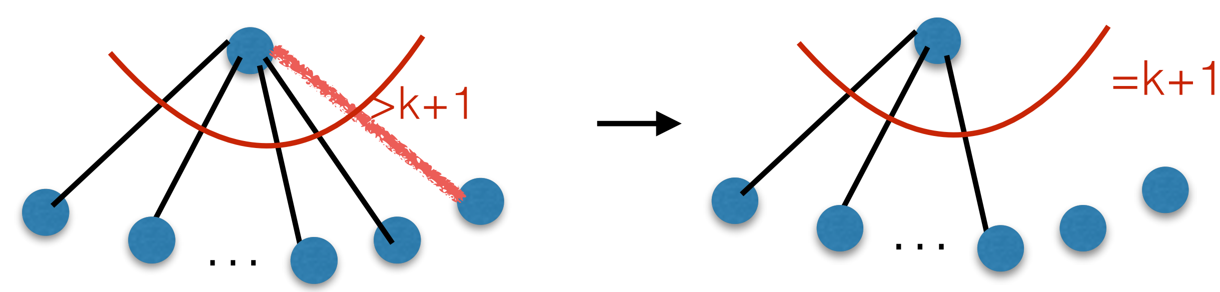

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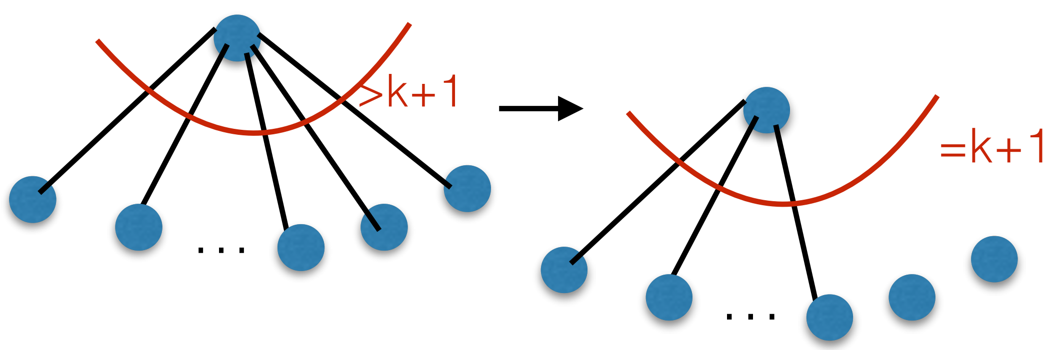

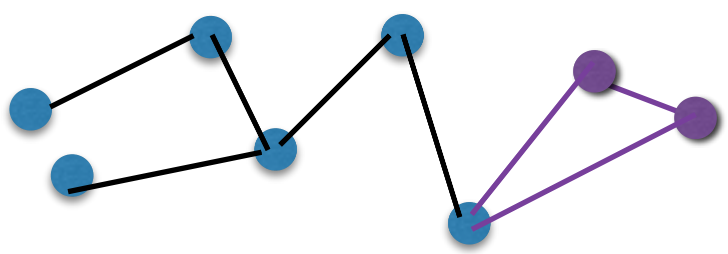

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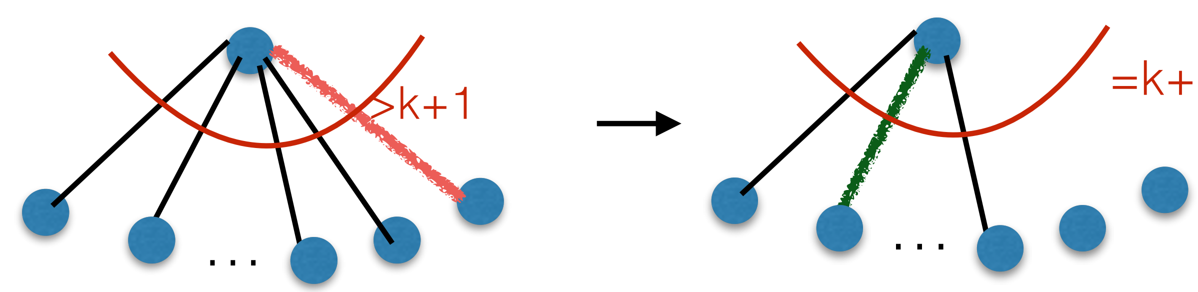

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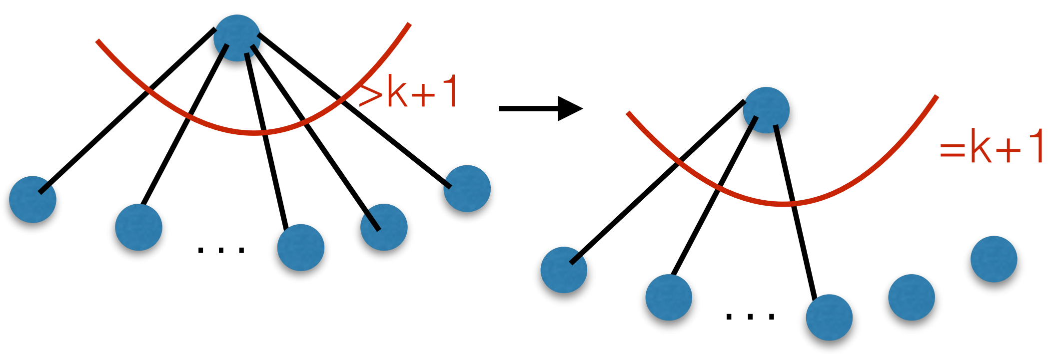

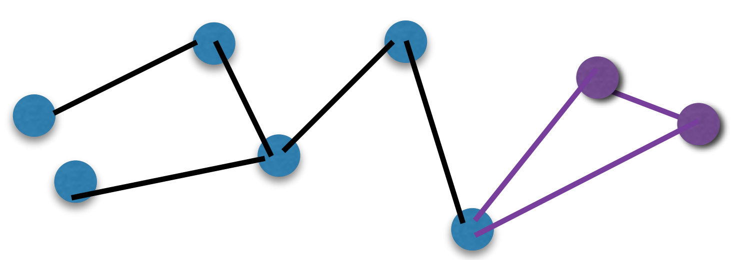

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Let X be a k size subset of $V(G_{i+1})$. If X is not a V.C of G_i, then X is not a V.C of G_{i+1}

Proof: Case 2: $G_i \rightarrow G_{i+1}$.

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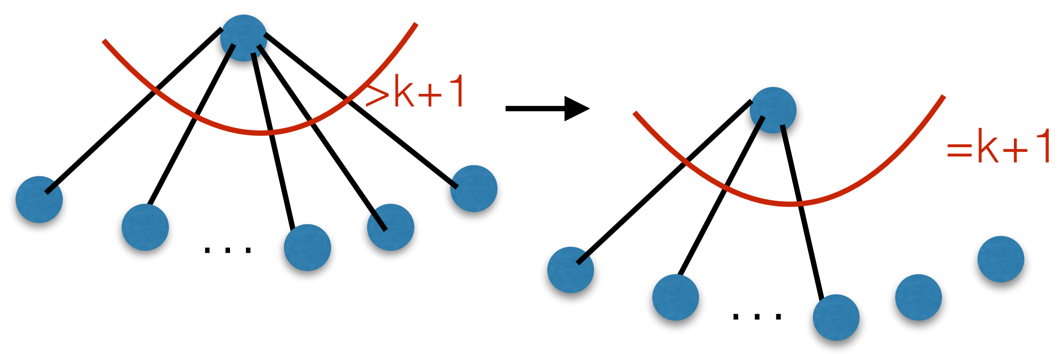
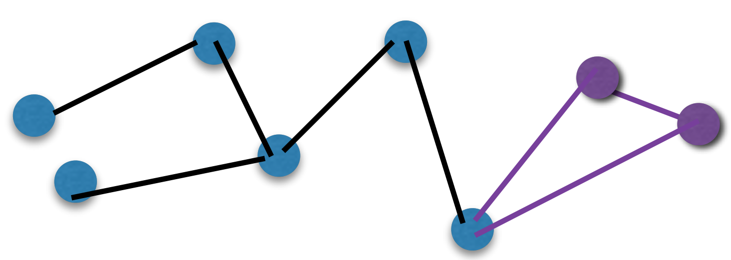
A Simple Kernel for VC





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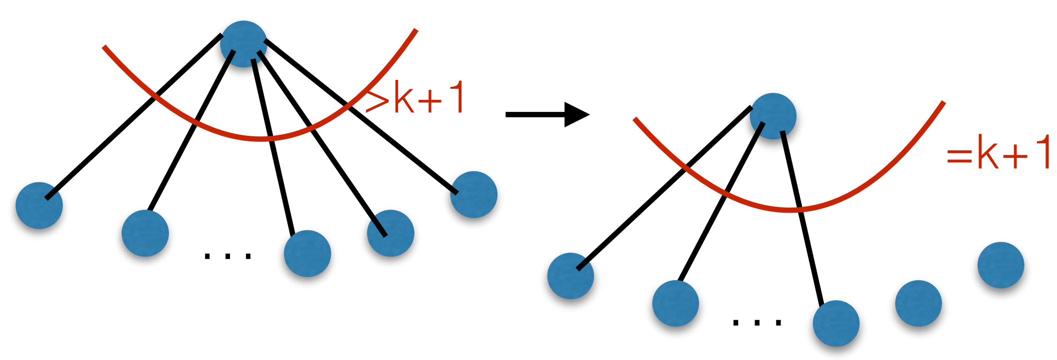

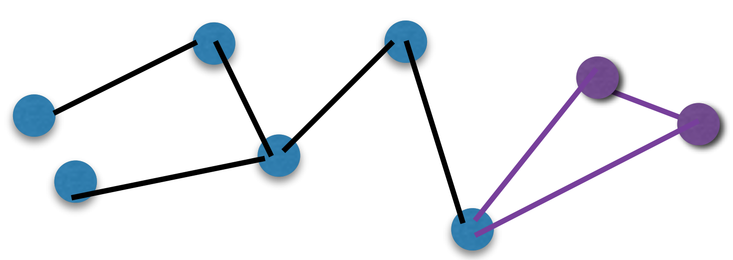

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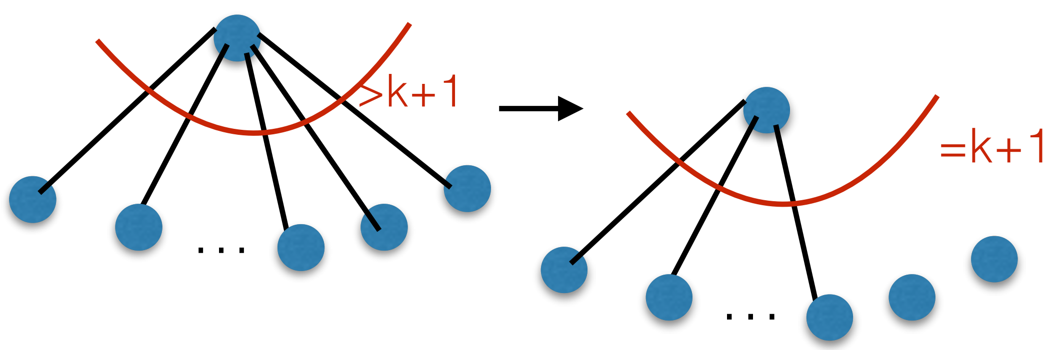
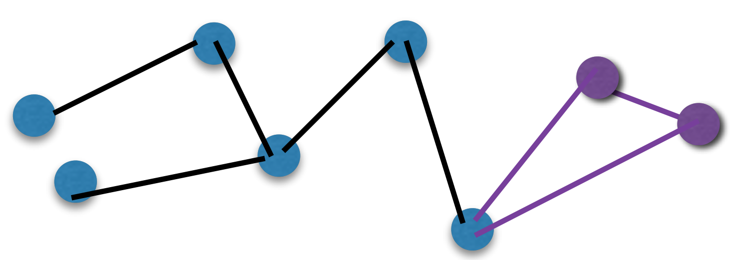
Let e be an edge in G_i not covered by X. If e is in $E(G_{i+1})$, then we are done.

Otherwise, we have deleted some edges. That is, $|E(G_{i+1})|=k(k+1)+1$ and max degree of G_{i+1} is k+1.

A Simple Kernel for VC





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Otherwise, we have deleted some edges. That is, $|E(G_{i+1})|=k(k+1)+1$ and max degree of G_{i+1} is $k+1$. So X cover at most $k(k+1)$ edges. One edge will be left uncovered. ■

Summary of VC kernel

(G,k)

$\downarrow$

Polytime Algo

$\downarrow$

$(H \leq G, k)$

- For any k size set X, if X is not a V.C of G, then X is not a V.C of H

Summery of VC kernel

(G,k)

Polytime Algo

$(H \leq G, k)$

- For any k size set X, if X is not a V.C of G, then X is not a V.C of H

$(E(G),k)$

Polytime Algo

$F=E(H) \subseteq E(G)$

- For any k size set X, if there is an edge {u,v} in E(G) such that $X \cap \{u,v\}=\emptyset$ then there is an edge {u′,v′} in F such that $X \cap \{u',v'\}=\emptyset$

Summery of VC kernel

(G,k)

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F is called k-representative family of E(G)

Representative Family

- $E \subseteq \binom{V}{2}$, where V is a set.
- k is a positive integer
- A subfamily $F \subseteq E$ is called a k-representative family if:

for any k-size set X if there is a set $Z \in E$ s.t $X \cap Z = \emptyset$,
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$\|F\|$	Run time
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Proof: The V.C kernel we have seen.

Representative Family
(for family of large subsets)

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$\lvert F \rvert$	Run time	Ref.
$\binom{k+p}{p}$	$O(\binom{k+p}{p}^{w-1} \lvert E \rvert)$	[Fomin et al. 2013]

5-Hitting Set

Input : A family $E \subseteq \binom{U}{5}$ of a set U and an integer k

Qn: Is there a k-size subset of U which hits all set in E

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(E, k) is Yes instance $\Rightarrow$ (F, k) is Yes instance.

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contradiction!

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in S, we get a solution S' s.t $|S' \cap F| > |S \cap F|$.

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This implies there is $Y' \in F$ s.t $X \cap Y' = \emptyset$. Then by replacing Y with Y' in S, we get a solution S' s.t $|S' \cap F| > |S \cap F|$. Contradiction!

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Output (F , k).

Proof: ($\Longleftarrow$)

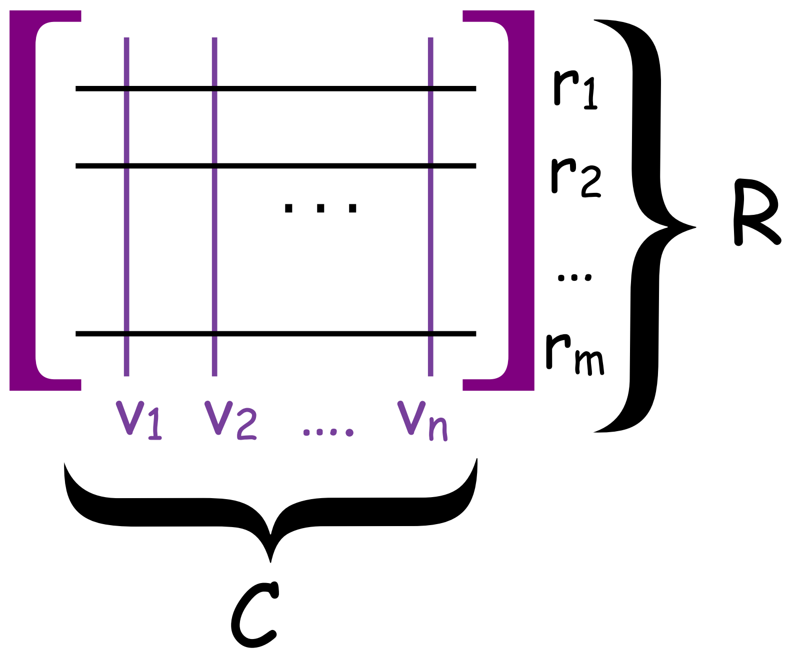
F is a subset of E

Generalization to Matrices

Definitions from Linear Algebra

$$\underbrace{\begin{array}{c} \begin{bmatrix} | & | & \cdots & | \\ | & | & & | \\ | & | & & | \end{bmatrix} \\ v_1 \quad v_2 \quad .... \quad v_n \end{array}}_{C}$$

Definitions from Linear Algebra



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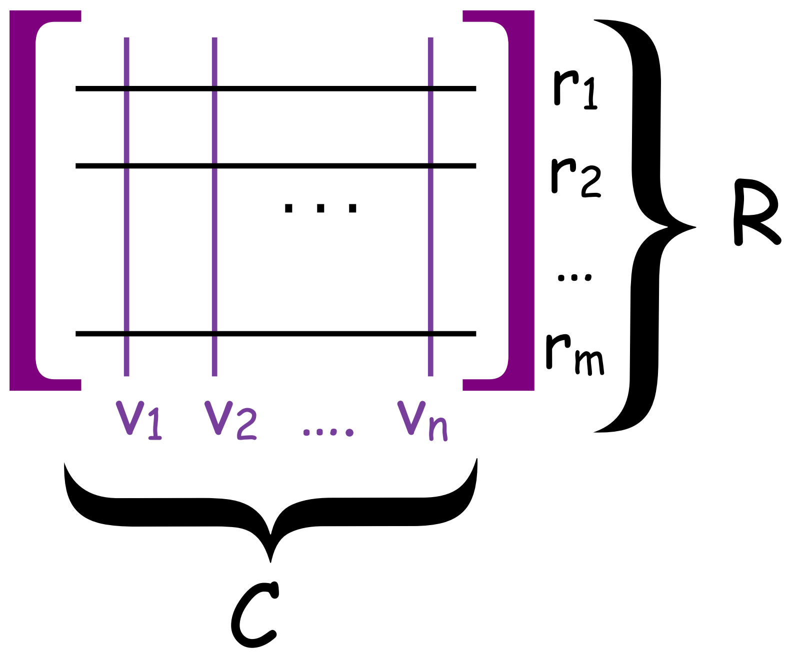

A set of vectors $v_1,\dots,v_t$ is linearly independent if there is no scalars $a_1,\dots,a_t$, not all equal to 0 such that

$$\sum a_i v_i = 0$$

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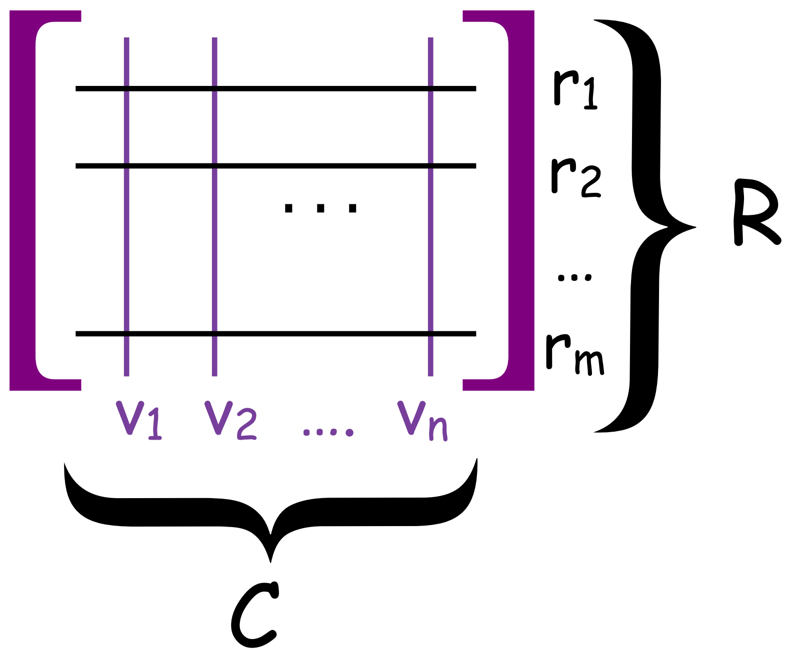

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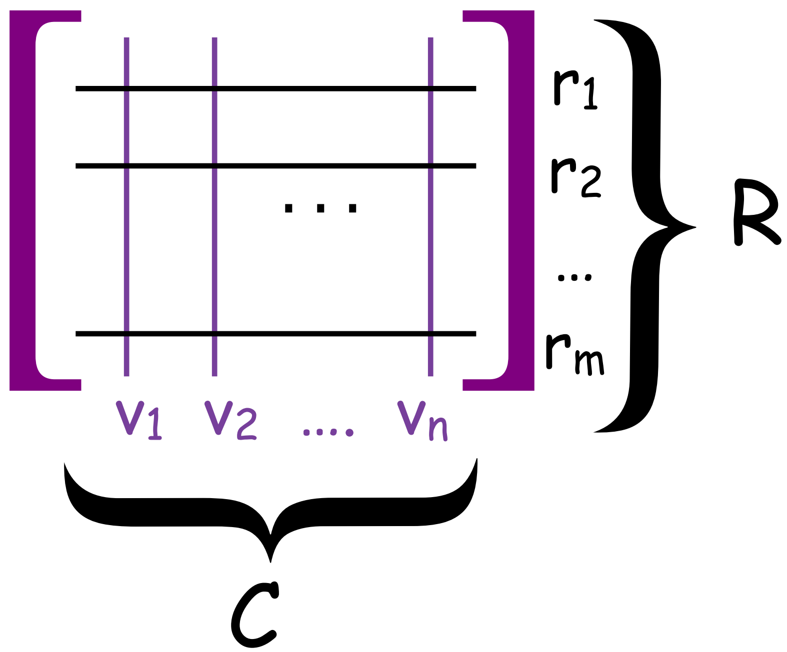
- Basis of C is a set of maximum no. of L.I vectors in C
- rank(C) = max. no. L.I vectors in C = size of a basis of C

Definitions from Linear Algebra



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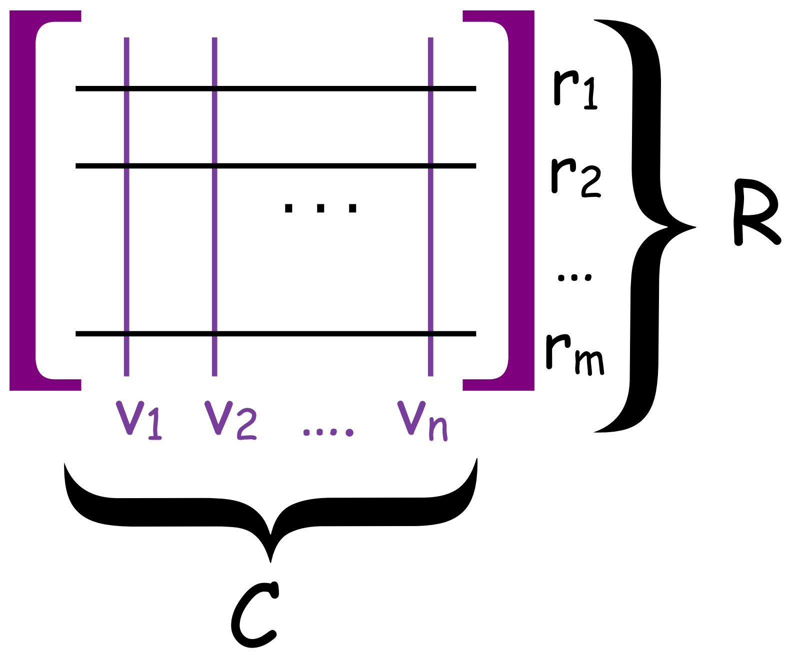
- rank(C) = rank(R)

Definitions from Linear Algebra



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- span(C) = set of all vectors which are linear combinations of C

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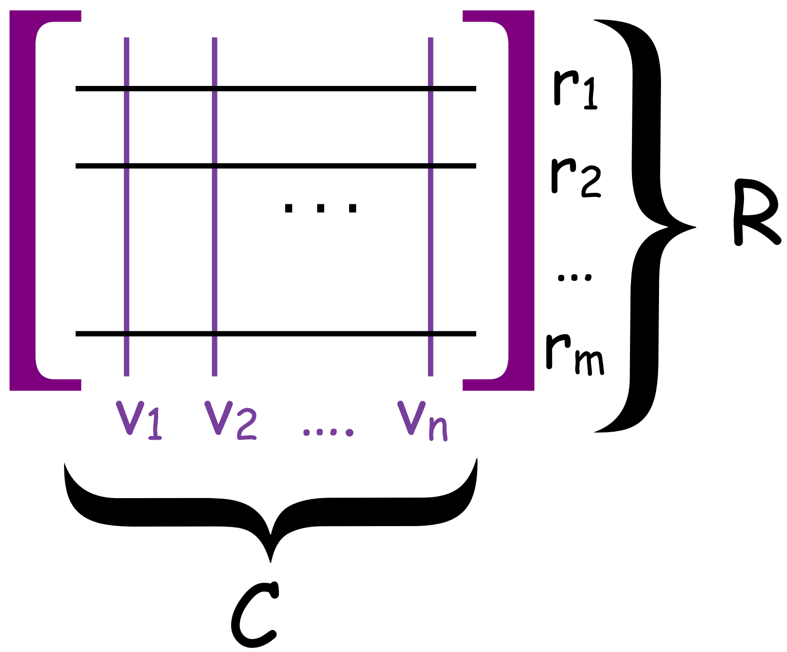

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- rank(span(C))=rank(C)

Representative Family on Matrices

$$\underbrace{\begin{array}{c}\begin{bmatrix} \vert & \vert & \cdots & \vert \\ \vert & \vert & & \vert \end{bmatrix} \\ v_1 \quad v_2 \quad \ldots \quad v_n\end{array}}_{V}$$

- E := a family of subsets of V, where each set size is p
- k is a postive integer

- A subfamily $F \subseteq E$ is called a k-representative family if:

for any k-size set X if there is a set $Z \in E$ s.t $X \cup Z$ is L.I, then there is a set Z' in F s.t $X \cup Z'$ is L.I.

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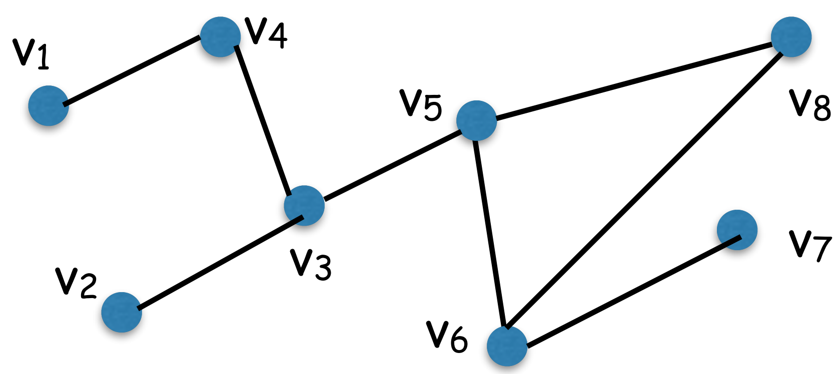
- A s

$\lvert F\rvert$	Run time	Ref.
$\binom{k+p}{p}$	$\binom{k+p}{p}^{w-1} \lvert E\rvert\, n^{O(1)}$	[Lokshtanov et al. 2013]

for any k-size set X if there is a set Z∈E s.t X ∪ Z is L.I, then there is a set Z′ in F s.t X ∪ Z′ is L.I.

Rank Vertex Cover



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

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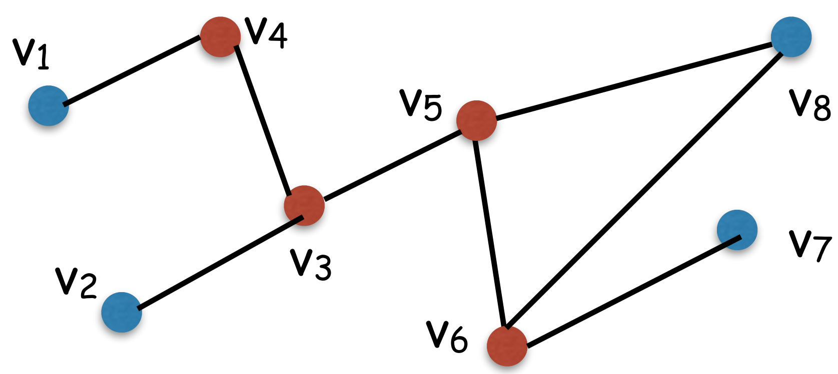

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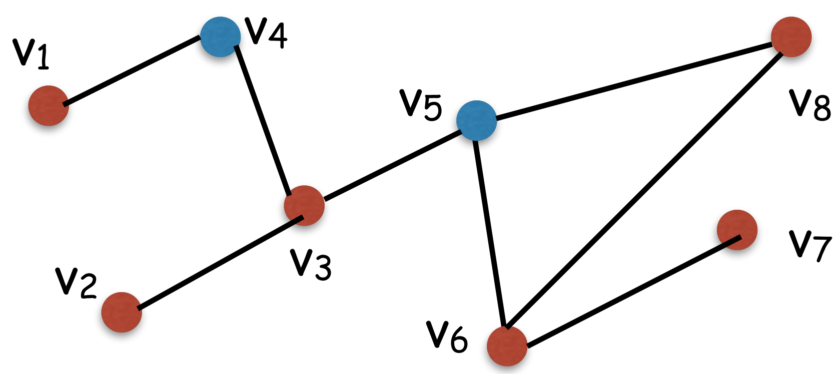
rank($\{v_3, v_4, v_5, v_6\}$) = Max no. L.I vectors in it = 4

Rank Vertex Cover



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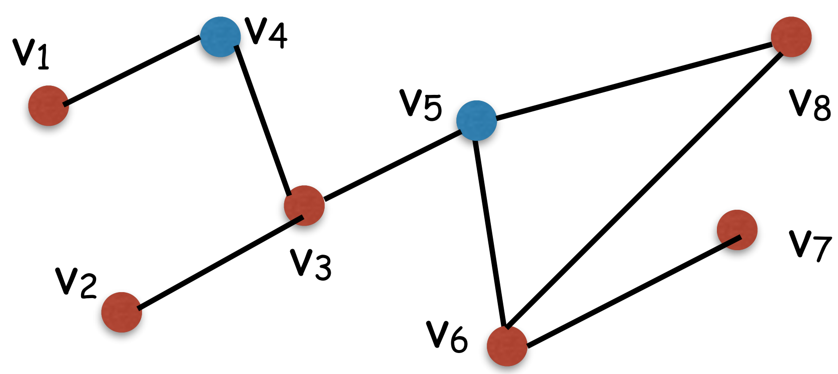
$\mathrm{rank}(\{v_3, v_4, v_5, v_6\})$ = Max no. L.I vectors in it = 4

$\mathrm{rank}(\{v_1, v_2, v_3, v_6, v_7, v_8\})$ = Max no. L.I vectors in it = 2

Rank Vertex Cover



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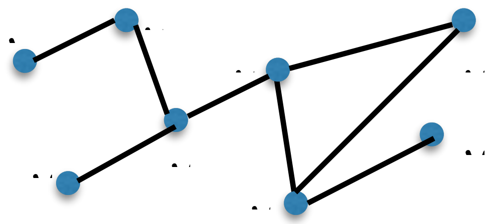
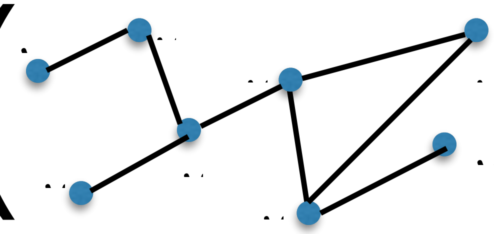
rank($\{v_3,v_4,v_5,v_6\}$) = Max no. L.I vectors in it = 4

rank($\{v_1,v_2,v_3,v_6,v_7,v_8\}$) = Max no. L.I vectors in it = 2

Input: A graph G, a matrix M, and an integer k

Question : Is there a vertex cover of rank k

V.C ≤ Rank V.C

( , k) ≤ ( , k , $\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdots & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$)

- G has a V.C of size k iff (G,M) has V.C of rank k

Kernel for Rank Vertex Cover

$$\left(G, k, M = \begin{array}{c} \begin{bmatrix} \vert & \vert & \cdots & \vert \\ \vert & \vert & & \vert \\ \vert & \vert & & \vert \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & \dots. & v_n \end{matrix} \end{array} \right)$$

Kernel for Rank Vertex Cover

$$\left(G, k, M = \begin{array}{c} \begin{bmatrix} \Big| & \Big| & \cdots & \Big| \end{bmatrix} \\ v_1 \;\; v_2 \;\; \ldots \;\; v_n \end{array} \right)$$

$$Q = \begin{array}{c} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \\ u \qquad\quad u' \end{array}$$

Kernel for Rank Vertex Cover

$$\left(G, k, M = \begin{array}{c} \begin{bmatrix} \mid & \mid & \cdots & \mid \end{bmatrix} \\ v_1 \;\; v_2 \;\; \ldots \;\; v_n \end{array} \right)$$

For any edge $\{u,v\}$ create a set $\{u,v'\}$ of two vectors in Q

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Kernel for Rank Vertex Cover

$$\left(G, k, M = \begin{array}{c} \left[\begin{array}{cccc} | & | & \cdots & | \end{array} \right] \\ \begin{array}{cccc} v_1 & v_2 & \dots & v_n \end{array} \end{array} \right)$$

For any edge {u,v} create a set {u,v'} of two vectors in Q

$$Q = \begin{array}{c} \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \\ \begin{array}{cc} u & u' \end{array} \end{array}$$

E is the collection of sets of vectors created

Kernel for Rank Vertex Cover

$$Q = \begin{array}{cc} u & u' \\ \end{array}$$
$$Q = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

E={{u,v'} : {u,v} in E(G)}

Kernel for Rank Vertex Cover

$$Q = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} u & u' \\ \end{array}$$

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- Compute 2k-representative family F of E ($|F|=O(k^2)$)

Kernel for Rank Vertex Cover

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- Compute 2k-representative family F of E ($|F|=O(k^2)$)
- Delete all edges which are not part of F in G
- Delete isolated vertices from G and M

Kernel for Rank Vertex Cover

$$Q = \begin{array}{cc} u & u' \\ \end{array}$$
$$Q = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

$E=\{\{u,v'\} : \{u,v\} \text{ in } E(G)\}$

- Compute 2k-representative family F of E ($|F|=O(k^2)$)
- Delete all edges which are not part of F in G
- Delete isolated vertices from G and M
- Call the new instance (G',M',k)

Kernel for Rank Vertex Cover

$$Q = \begin{array}{cc} u & u' \\ \end{array}$$
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- Compute 2k-representative family F of E ($|F|=O(k^2)$)
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- Delete isolated vertices from G and M
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- Size of G' is $O(k^2)$
- No. of columns in M' is $O(k^2)$

Kernel for Rank Vertex Cover

$$Q = \begin{array}{cc} u & u' \\ \end{array}$$
$$Q= \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$$

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- Compute 2k-representative family F of E (|F|=$O(k^2)$)
- Delete all edges which are not part of F in G
- Delete isolated vertices from G and M
- Call the new instance (G',M',k)
- Size of G' is $O(k^2)$
- No. of columns in M' is $O(k^2)$
- The number of no. of rows in M' is not bounded

Kernel for Rank Vertex Cover

$$Q = \begin{array}{c} \begin{array}{cc} u & u' \end{array} \\ \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \end{array}$$

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- Call the new instance (G',M',k)
- Size of G' is $O(k^2)$
- No. of columns in M' is $O(k^2)$
- The number of no. of rows in M' is not bounded

Just see the correctness proof of this reduction

Proof: Forward direction

- G' is a subgraph of G
- M' is obtained by deleting some columns from M
- If X is V.C of rank k in G, then $X \cap V(G')$ is a V.C of rank k in G'

Proof: Reverse direction

- Let X' is a V.C of G' and rank(X')=k

Proof: Reverse direction

- Let X' is a $V.C$ of G' and $rank(X')=k$
- We claim $X=span(X') \cap M$ is a $V.C$ of G.

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Proof: Reverse direction

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- Let B be a basis of X

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$$\begin{array}{cc} & u \\ Q= & \begin{bmatrix} B & 0 & \cdots \\ \cdots & & \\ 0 & B & \end{bmatrix} \\ & v' \end{array}$$

B_1 B_2

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- $B_1 \cup \{u\}$ is L.I
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$$
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\end{array}
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- $B \cup \{w\}$, $B \cup \{z\}$ are L.I
- That is, X' will not cover the edge {w,z} in G'

Proof: Reverse direction

- Let X′ is a V.C of G′ and rank(X′)=k
- We claim X=span(X′)∩M is a V.C of G.
- Suppose not. Let {u,v} be an edge not covered by X
- Let B be a basis of X
- |B|=k
- B ∪ {u}, B ∪ {v} are L.I

$$Q=\begin{array}{c} \begin{array}{cc} u & \\ \end{array} \\ \begin{bmatrix} B & 0 \\ \cdots & \cdots \\ 0 & B \end{bmatrix} \\ \begin{array}{cc} & v' \end{array} \end{array}$$

B_1 B_2

- B_1 ∪ {u} is L.I
- B_2 ∪ {v′} is L.I
- B_1 ∪ B_2 ∪ {u,v′} is L.I
- |B_1 ∪ B_2 |=2k
- From the def. of rep. family there is {w,z′} in E s.t. B_1∪B_2∪{w,z′} is L.I
- B ∪ {w}, B ∪ {z} are L.I
- That is, X′ will not cover the edge {w,z} in G′

Contradiction!

Bound on no. of rows

- We got (G',M',k), where size of G' and no. columns in M' is $O(k^2)$.

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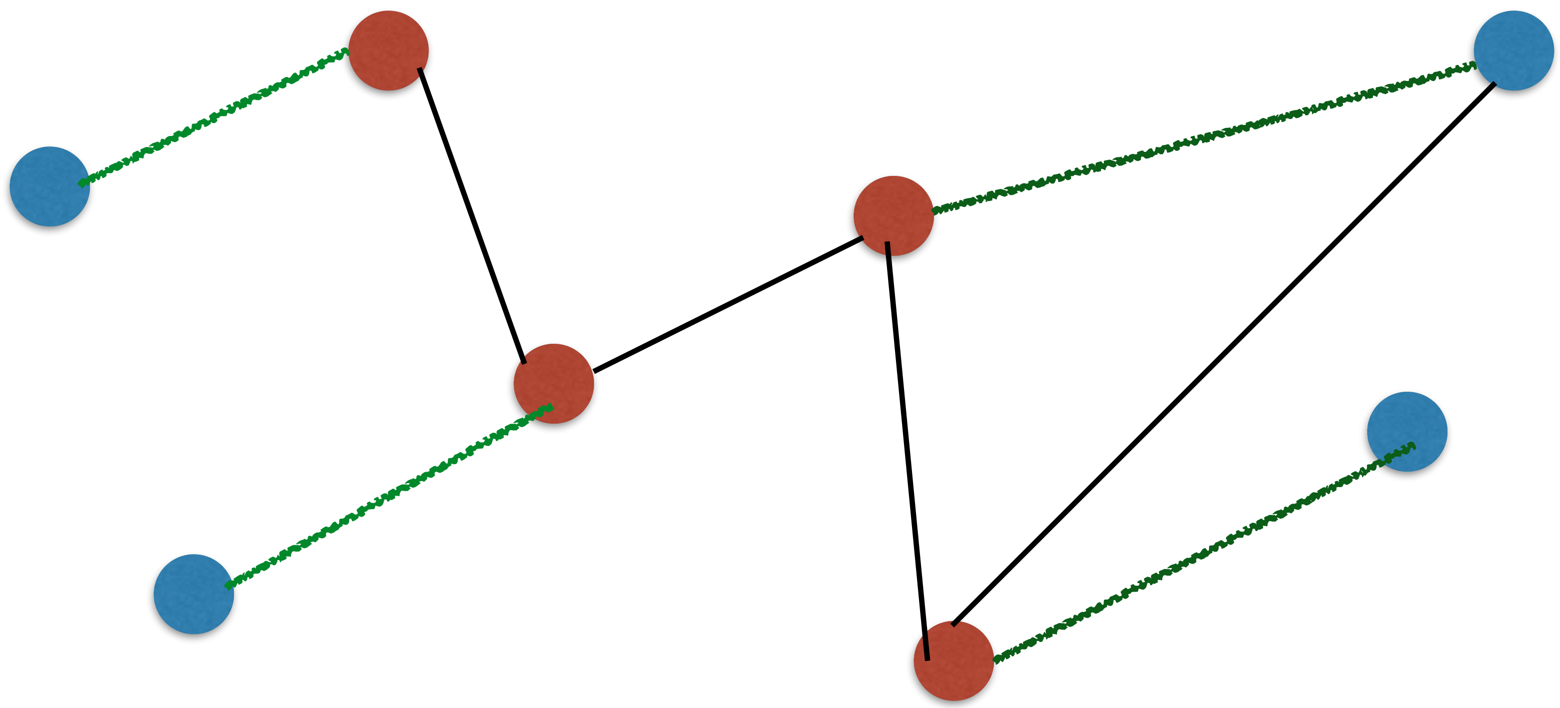
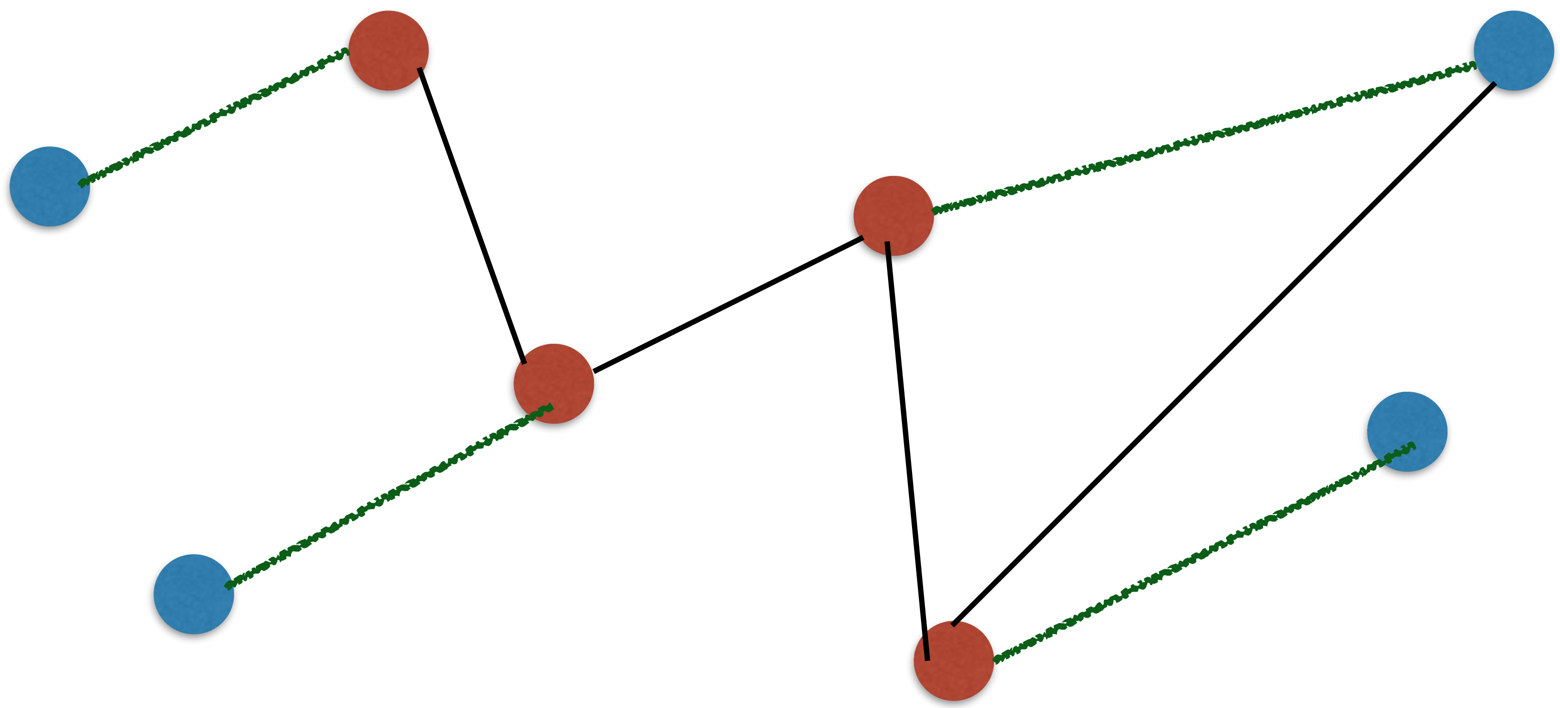
Proof omitted
(uses elementary operations)

V.C above Max matching



V.C above Max matching



- Size of a V.C is at least the size a max. matching.
- Max. matching can be computed in polynomial time

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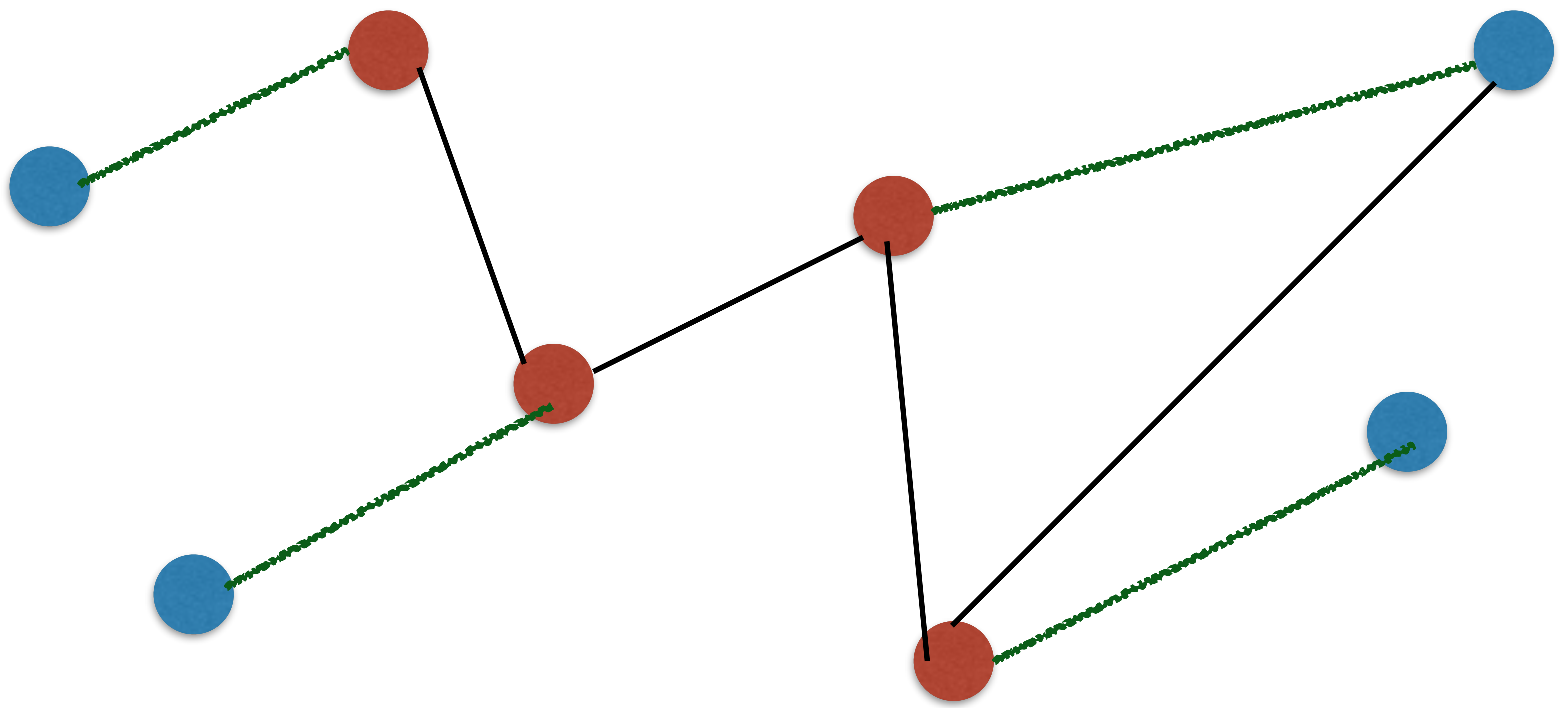
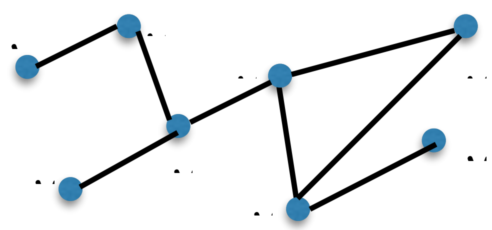
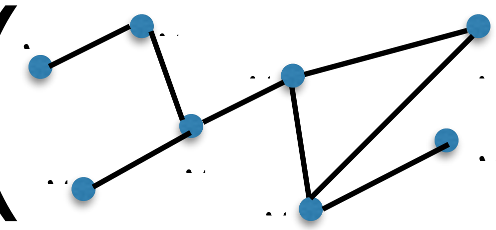


- Size of a V.C is at least the size a max. matching.
- Max. matching can be computed in polynomial time

Input: A graph G and an integer k

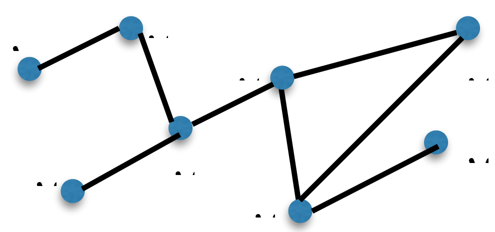
Question : Is there a vertex cover of size mm(G)+k

V.C above mm ≤ Rank V.C

( , k) $\leq$ ( , mm+k , $\begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \\ & & & & 1 \end{bmatrix}$)

- G has a V.C of size mm(G)+k iff (G,M) has V.C of rank mm(G)+k

Rank Reduction

( , mm+k , $\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \cdots & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$)

Rand. polynomial Time

$\left(G', k' = O(k^{3/2}),\ M' \right)$

Each entry is on poly(k) bits

How to get a Kernel for V.C above MM

(G,k) of V.C above MM

$\downarrow$

($G,mm+k,I_n$) of Rank V.C

How to get a Kernel for V.C above MM

(G,k) of V.C above MM

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(G,mm+k,I_n) of Rank V.C

$\downarrow$

Matrix entry bounded in poly(k) (G',O($k^{3/2}$),M') of Rank V.C

How to get a Kernel for V.C above MM

(G,k) of V.C above MM

↓

$(G,mm+k,I_n)$ of Rank V.C

↓

Matrix entry bounded in poly(k) $(G',O(k^{3/2}),M')$ of Rank V.C

↓

Size bounded in poly(k) $(G'',O(k^{3/2}),M'')$ of Rank V.C

How to get a Kernel for V.C above MM

(G,k) of V.C above MM

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↓

Size bounded in poly(k) (H, k'') of V.C above MM

How to get a Kernel for V.C above MM

(G,k) of V.C above MM

↓

$(G, mm+k, I_n)$ of Rank V.C

↓

Matrix entry bounded in poly(k) $(G', O(k^{3/2}), M')$ of Rank V.C

↓

Size bounded in poly(k) $(G'', O(k^{3/2}), M'')$ of Rank V.C in NP

↓

Size bounded in poly(k) (H, k'') of V.C above MM is NP-hard

How to get a Kernel for V.C above MM

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(G,mm+k,I_n) of Rank V.C

Matrix entry bounded in poly(k) (G',O($k^{3/2}$),M') of Rank V.C

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in NP

Size bounded in poly(k) (H,k'') of V.C above MM

is NP-hard

Co-loop in a matrix

$$M = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

- Co-loop is a column vector which is part of any basis.

Rank Reduction

$(G, mm+k, I_n)$ of Rank V.C

↓

$(G', O(k^{3/2}), M')$ of Rank V.C

(H, p, N) of Rank V.C and a co-loop v in N

↓ (in Rand. Polynomial time)

$(H \backslash u, p-2, N')$ of Rank V.C

- Co-loops in $(N \setminus N(v)) \subseteq$ Co-loops in N'

Rank Reduction

(H, p, N) of Rank V.C and a co-loop u in N

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V.C above MM

(G,k) → Rand. poly time algorithm [Misra et al., 2011] → Yes/No; a V.C of size $mm+ck^{3/2}$

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V.C above MM

(G,k)

Rand. poly time algorithm [Misra et al., 2011]

Yes/No

a V.C of size $mm+ck^{3/2}$

an Independent set of size $n-(mm+ck^{3/2})$

Rank Reduction

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$(G', O(k^{3/2}), M')$ of Rank V.C

- Recall that $(G, mm+k, I_n)$ is the input instance of Rank V.C

Rank Reduction

(H, p, N) of Rank V.C and a co-loop u in N

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- Recall that $(G, mm+k, I_n)$ is the input instance of Rank V.C
- Run the algorithm by Misra et al., 2011 on (G,k) and if it outputs Yes/No, we are done.

Rank Reduction

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- We apply the above reduction rule one by one on S.

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$$\text{rank}(M')=n-2|S|=n-2(n-mm-ck^{3/2}) \leq -n+2mm+2ck^{3/2} \leq 2ck^{3/2}$$

Conclusion

Kernelization: V.C above MM, V.C above LP, Almost 2SAT, Multiway Cut with deletable terminals, Subset FVS, etc
Open problems: deterministic polynomial kernels for the above problems ?

FPT: k-Matroid Parity, k-Path, k-Tree, Connectivity problems on graphs of bounded tree-width, Long Cycle, k-MLD, etc

Exact Exponential Time algorithms : Min. Equivalent digraph, Minimum Weight λ-connected Spanning Subgraph.

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Thank you for your attention!