Linear kernel for planar Dominating Set

Input: (G, k), G is planar

Ouput: (G', k') s.t.

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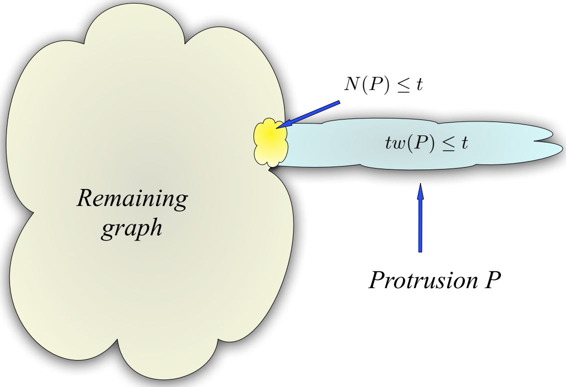
- $|G| = \mathcal{O}(k)$.

Tool

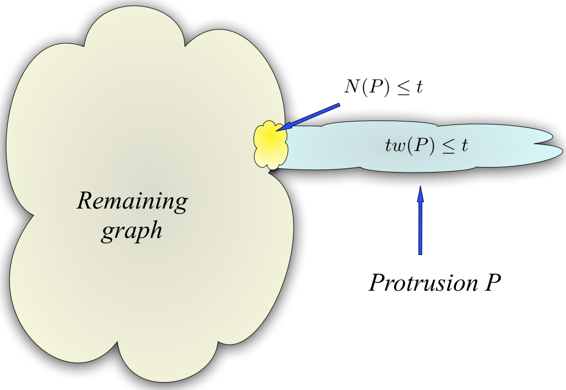
Protrusion in planar graph



Tool

Reducing Protrusion

Reduce the number of vertices in P down to $\mathcal{O}(|OPT \cap P|)$.



Protrusion Reduction Rule

- For every partial solution D of $N(P)$ compute a minimum extension of D to $P \cup N(P)$. (Partial solution: in dominating set, is dominated and not dominated.) We have at most 3^t partial solutions for each P .

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Protrusion Reduction Rule

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- Delete edges with both unmarked endpoints. (If there is an optimal solution in G, then there is an optimal solution using only marked vertices.)
- Delete isolated vertices.

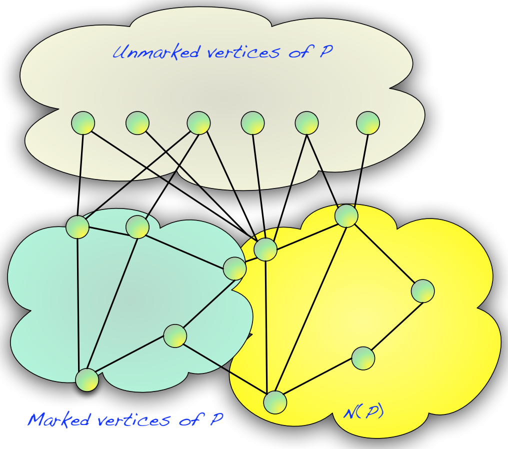
Protrusion Reduction Rule

How many marked vertices in P?

- $3^t(|OPT(G) \cap P| + t)$

Reduction Rule for unmarked vertices

How many unmarked vertices in P?



Planar Bipartite Lemma

Lemma (Planar Bipartite Lemma)

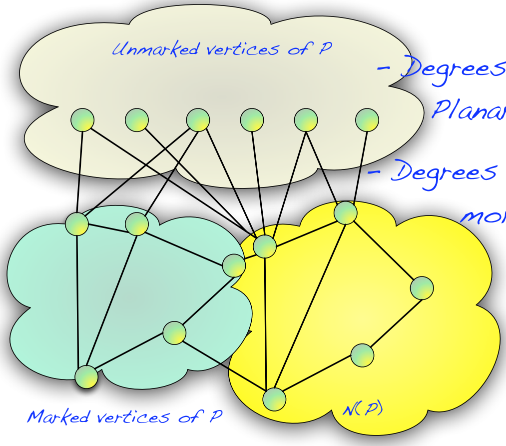
Let G be a simple bipartite graph with a bipartition (A, B) such that every vertex $v \in B$ has at least three neighbors in A. Then $|B| \leq 2|A|$

Proof: Euler's formula $m = n + f - 2$.

Reduction Rule for unmarked vertices

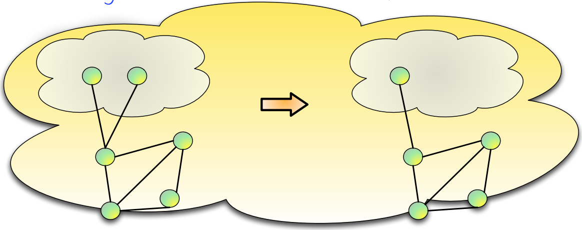
How many unmarked vertices in P?

- Degrees >= 3: constant (again Planar Bipartite Lemma)
- Degrees 1 and 2 can be many, need more reduction rules

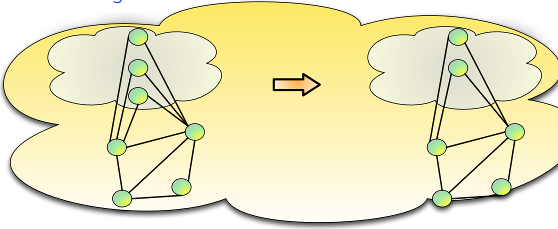


Reduction Rule for unmarked vertices

Degree-1 Reduction Rule for unmarked vertices



Degree-2 Reduction Rule for unmarked vertices



Reduction Rule for unmarked vertices

After application of Reduction Rules, what remains from P has $3^t(|OPT(G) \cap P| + t)$ marked and $\mathcal{O}(3^t(|OPT(G) \cap P| + t))$ unmarked vertices.

Strategy

- Decompose G into protrusions

Strategy

- Decompose G into protrusions
- Use protrusion reduction rule

Decompositing G

Input: (G, k), G is planar (and connected)

Compute a connected dominating set S of size $\mathcal{O}(k)$.

Claim

Every connected component of $G - S$ is an outerplanar graph.

Decomposing G

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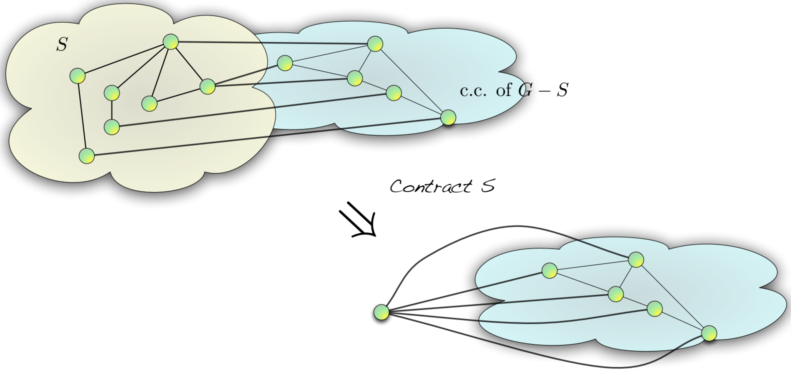
Every connected component of $G - S$ is an outerplanar graph.

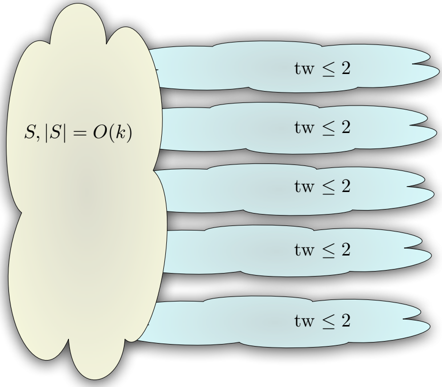
Proof:



Decomposing G

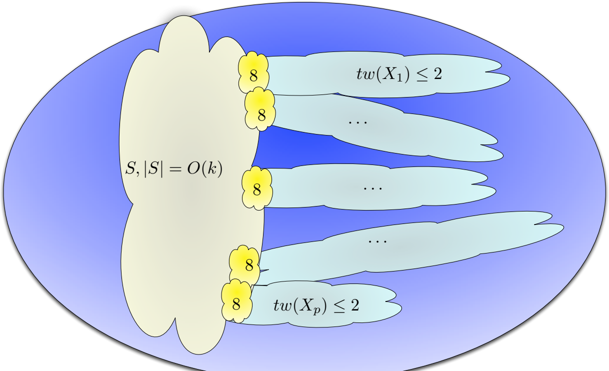
We have set S of size $\mathcal{O}(k)$ and every connected component of $G - S$ is of treewidth at most 2.

But the neighborhoods of components can be large.

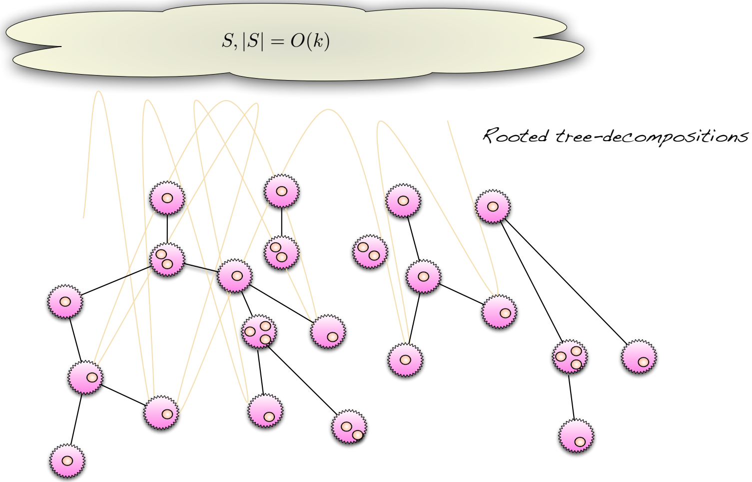


Decomposing G

We have set S of size $\mathcal{O}(k)$ and every connected component of $G - S$ is of treewidth at most 2. We want (a little bit) more. We want to enhance S such that S is still of order $\mathcal{O}(k)$ but for every c.c. X of $G - S$, $|N_G(X)| \leq 8$.



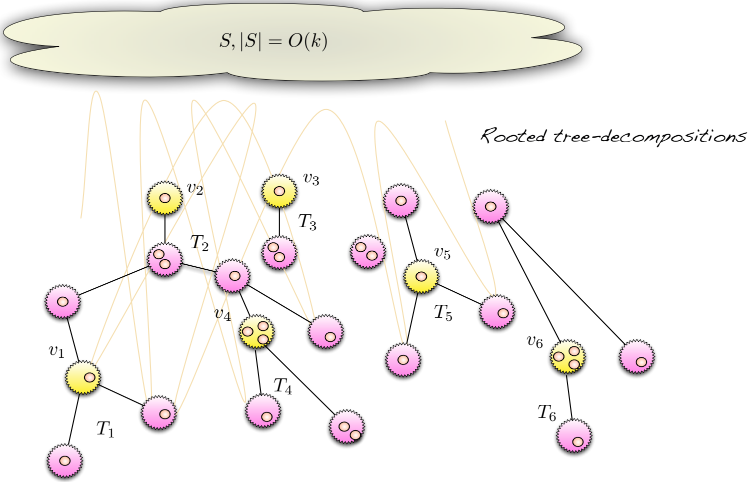
Cutting components



Marking step

- Set marked vertices $M = \emptyset$, $i = 1$
- While a component C of $G - (S \cup M)$ has ≥ 3 neighbors in S **do**
  - Let v_i be the leftmost bag in the tree decomposition T of C such that the vertices contained in bags of the subtree T_i rooted in v_i have ≥ 3 neighbors in S
  - Add to M vertices contained in v_i
  - $i := i + 1$, delete vertices contained in nodes of T_i from G.

Cutting components



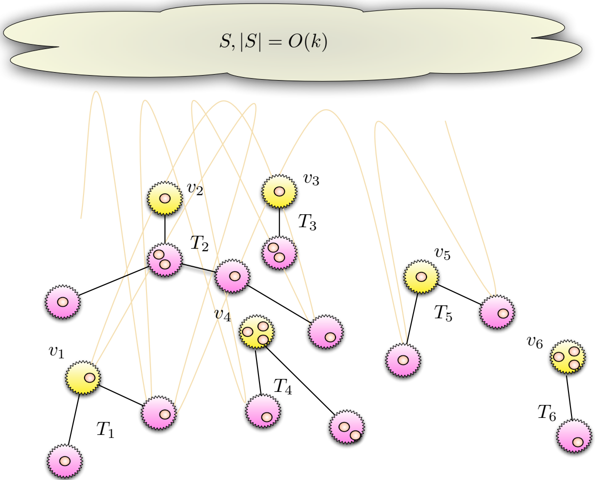
How many vertices are marked? Again

Lemma (Planar Bipartite Lemma)

Let G be a simple bipartite graph with a bipartition (A, B) such that every vertex $v \in B$ has at least three neighbors in A. Then $|B| \leq 2|A|$

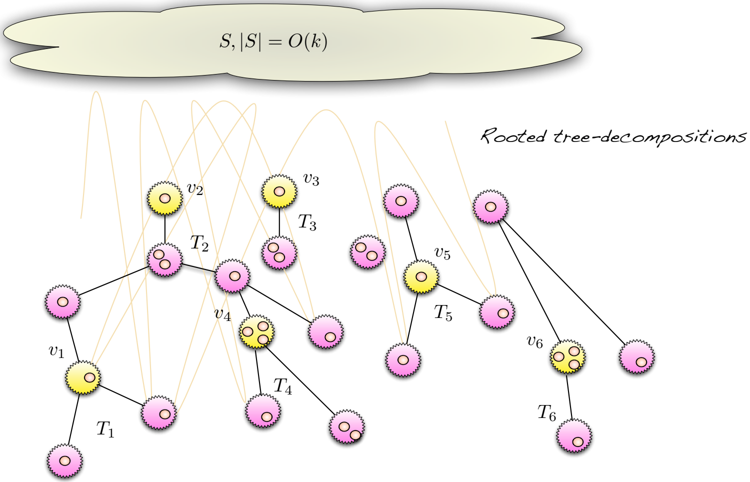
How many vertices are marked?

By Planar Bipartite Lemma, $i \leq 2|S|$. Each node v_i contains at most 3 vertices, hence $|M| \leq 6|S|$.



Cutting components

Every c.c. of $G - (S \cup M)$ sees at most 2 vertices in S.



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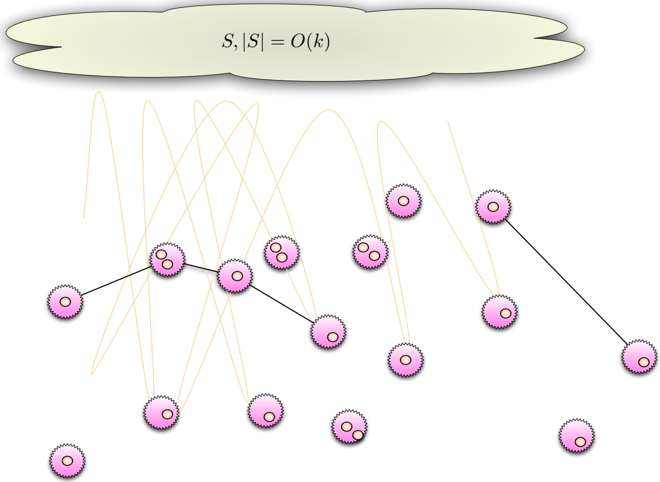

We are almost there!

Every c.c. of $G - (S \cup M)$ sees at most 2 vertices in S but can see many vertices in M.

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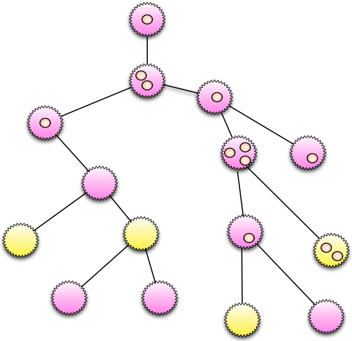
Every c.c. of $G - (S \cup M)$ sees at most 2 vertices in S but can see many vertices in M.

Mark more vertices!

Least common ancestor closure

While there are two marked nodes whose common ancestor is not marked, mark it.



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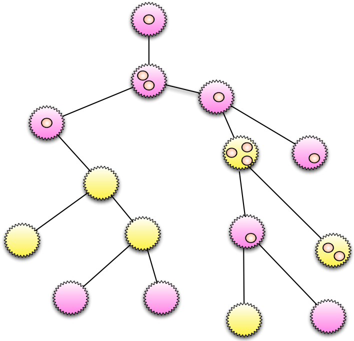

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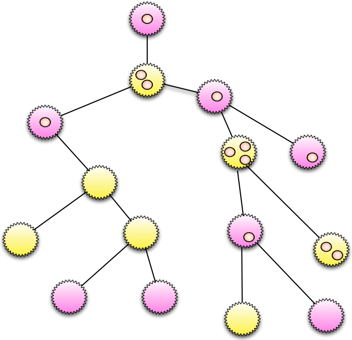

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- Put all marked vertices in M. The size of $S \cup M$ is now at most $13|S|$

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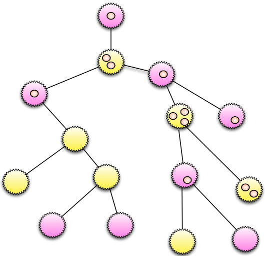
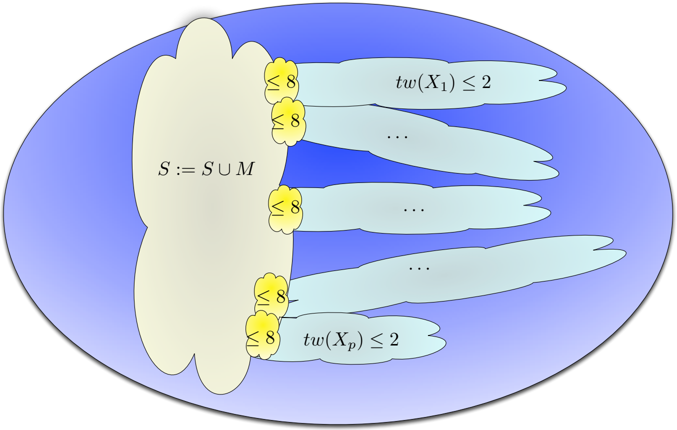
- Put all marked vertices in M. The size of $S \cup M$ is now at most $13|S|$
- Every connected component of $G - (S \cup M)$ sees at most 2 vertices in S and at most $2 \cdot 3 = 6$ vertices in M



Final decomposition



Different neighborhoods

Group components of $G - S$ into classes according to their neighborhoods in S.

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Group components of $G - S$ into classes according to their neighborhoods in S. How many classes?

- Number of different size-1 neighborhoods is at most $|S|$
- Number of different size-2 neighborhoods is at most $3|S|$ (Euler formula)
- Number of different size-≥ 3 neighborhoods is at most $2|S|$ (Planar Bipartite Lemma)

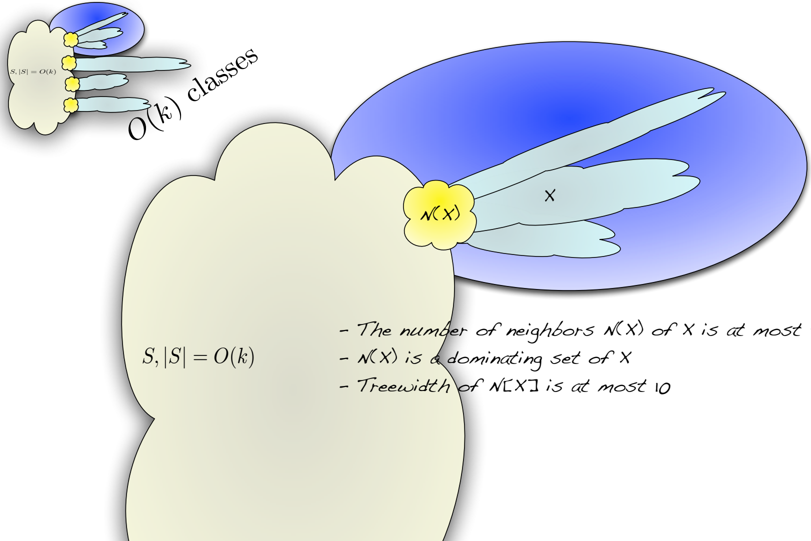
Different neighborhoods

Group components of $G - S$ into classes according to their neighborhoods in S. How many classes?

- Number of different size-1 neighborhoods is at most $|S|$
- Number of different size-2 neighborhoods is at most $3|S|$ (Euler formula)
- Number of different size-≥ 3 neighborhoods is at most $2|S|$ (Planar Bipartite Lemma)

In total, at most $6|S|$ classes.

Closer look at a class



- The number of neighbors N(X) of X is at most 8
- N(X) is a dominating set of X
- Treewidth of N[X] is at most 10

Reduction Rule

- Each class X is a protrusion, so we can apply Protrusion Reduction Rule

Reduction Rule

- Each class X is a protrusion, so we can apply Protrusion Reduction Rule
- Since for each X, $|OPT(G) \cap X| \leq 8$, each protrusion shrinks to constant size.

Recap of the algorithm

- Find a set S of size $O(k)$ s.t. $G - S$ is of constant treewidth

- Find a set M of size $O(k)$ s.t. every c.c. of $G-(S \cup M)$ sees constant number of vertices in $S \cup M$

- Group c.c. of $G-(S \cup M)$ into classes according to their neighborhoods

- Reduce each class to constant size

Where the property of a dominating set was important?

✓ - Find a set S of size $O(k)$ s.t. $G - S$ is of constant treewidth

- Find a set M of size $O(k)$ s.t. every c.c. of $G-(S \cup M)$ sees constant number of vertices in $S \cup M$

- Group c.c. of $G-(S \cup M)$ into classes according to their neighborhoods

✓ - Reduce each class to constant size

Abstraction of the property

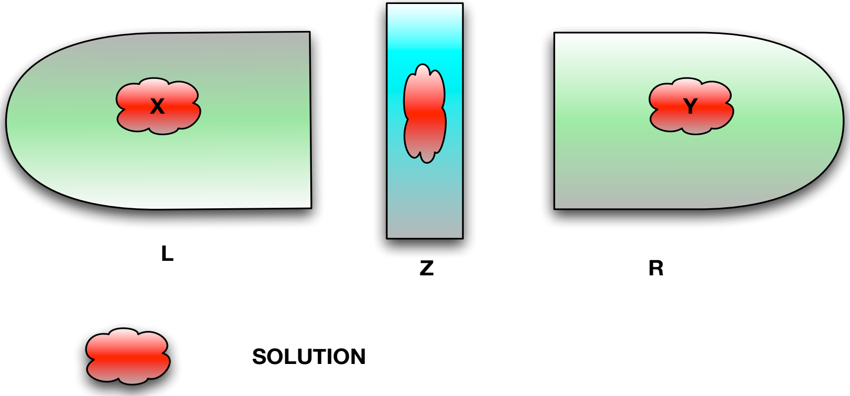
Parameterized problem (G,k) such that:

- If (G,k) is a YES instance, we can find a set S of size $O(k)$ s.t. $G - S$ is of constant treewidth

- We can reduce each class to constant size

WE HAVE LINEAR KERNEL!

Concept of Separability



$$|OPT(L \cup Z)| = |OPT(G) \cap L| \pm \mathcal{O}(|Z|)$$

$$|OPT(R \cup Z)| = |OPT(G) \cap R| \pm \mathcal{O}(|Z|)$$

Bidimensional Problems

- Minor Free Graph Class + Minor Bidimensionality

$$\implies \mathrm{tw}(\mathcal{O}(\sqrt{k}))$$

(Sublinear Treewidth Parameter Bound)

- APEX-Minor Free Graph Class + Contraction Bidimensionality

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- APEX-Minor Free Graph Class + Contraction Bidimensionality

$$\implies \mathrm{tw}(\mathcal{O}(\sqrt{k}))$$

- Separability + Sublinear Treewidth Parameter Bound

$$\implies \exists\, S \text{ of size } \mathcal{O}(k) \text{ such that } \quad \mathrm{tw}(G[V \setminus S]) \leq \eta$$

(η is a constant that only depends on the problem)

Treewidth modulator

Main Idea: All Bidimensional and Seperable problems are

"constant factor" related to:

Treewidth η-Modulator

Input: A graph $G = (V, E)$ and a positive integers k.

Parameter: k

Question: Does there exist a subset $F \subseteq V$ of size at most k such that $\mathrm{tw}(G[V \setminus F]) \leq \eta$?

on "planar like graphs".

Approximating treewidth modulator

Constant factor approximation:

FVF, Daniel Lokshtanov, Neeldhara Misra, Saket Saurabh

Planar F-Deletion: Approximation, Kernelization and Optimal FPT Algorithms. FOCS 2012: 470-479

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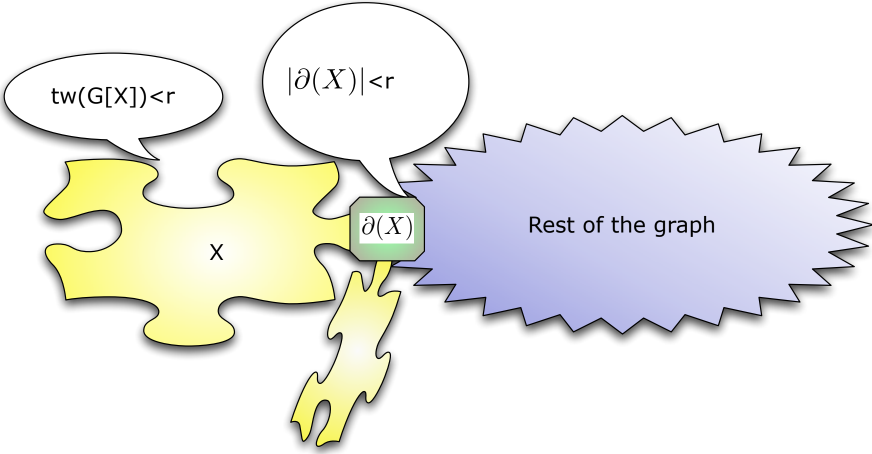
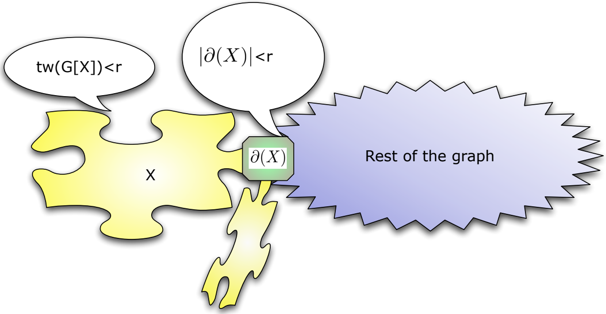
Thus for every instance (G, k) of a bidimensional separable problem on planar (apex-minor-free, minor-free) one can find in polynomial time a set S of size $\mathcal{O}(k)$ such that $G - S$ is of constant treewidth.

Reduction rules and protrusions



Protrusions



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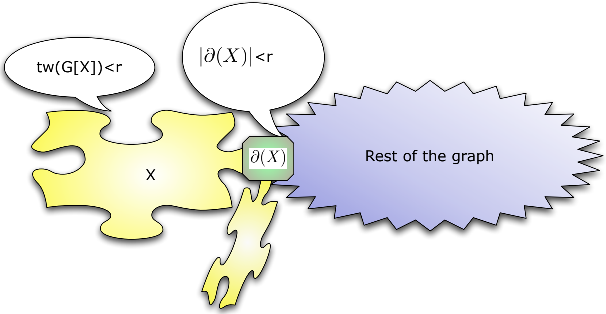
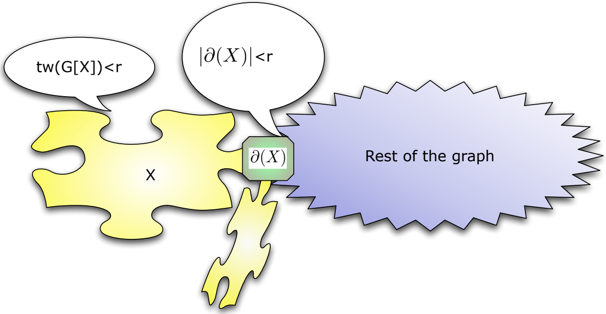


Idea to use this object in kernelization:

Bodlaender, FF, Lokshtanov, Penninkx, Saurabh, Thilikos

(Meta) Kernelization. FOCS 2009: 629-638

Protrusions



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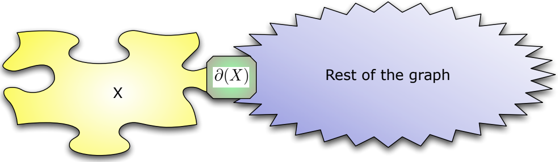
(Meta) Kernelization. FOCS 2009: 629-638 Used before in the

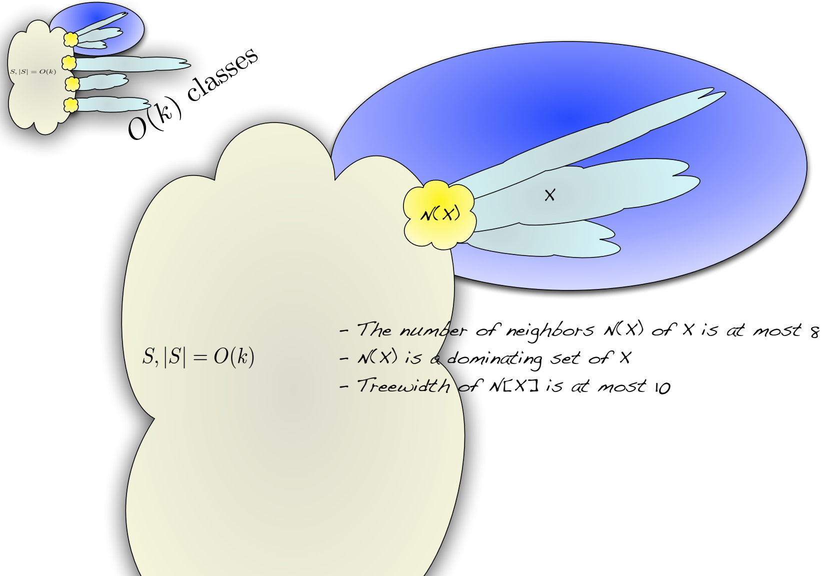
development of algorithms on graphs of bounded treewidth.

How protrusions work for parameterized problem Π

If the size of protrusion X is larger than some constant x (depending only on Π), it is possible to replace X by a protrusion X' of size $x' < x$ such that the solution for Π remains the "same" on the new graph.



In the dominating set example protrusions were the sets X formed by different classes of neighborhoods in S.



Protrusion replacer

A protrusion replacer is an algorithm (well, a sequence of algorithms, one for each r) that in polynomial time reduces each protrusion to the size $f(r)$.

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For DS we use $r \leq 8$.

Theorem

Every separable CMSO-Optimization problem has a protrusion replacer.

Putting things together

Theorem

Every bidimensional separable CMSO-Optimization problem has a linear kernel on planar graphs.

Putting things together

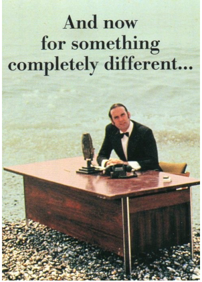
Theorem

Every bidimensional separable CMSO-Optimization problem has a linear kernel on planar graphs.

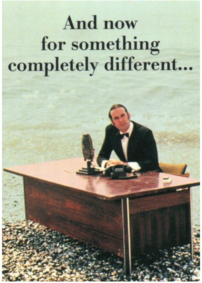
Where did we use planarity?

CMSO Bidimensional separable problems

Dominating Set, r-Dominating Set, Vertex Cover, Connected r-Dominating Set, Connected Vertex Cover, Vertex-Minor-Covering, Minimum Maximal Matching, Vertex-Subgraph-Covering, Clique-Transversal, Almost-Outerplanar, Feedback Vertex Set, Cycle Domination, Edge Dominating Set, Independent Set, Induced d-Degree Subgraph, r-Scattered Set, Induced Matching, Triangle Packing, Cycle Packing ...



- School on Recent Advances in Parameterized Complexity will be held on December 3-7, 2017, Tel Aviv, Israel.



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- PhD position in Bergen (send me email)