Lower bounds for polynomial kernelization

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Parameterized Complexity Summer School

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Kernelization — recap

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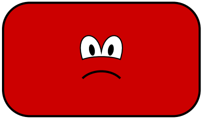

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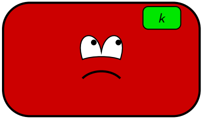

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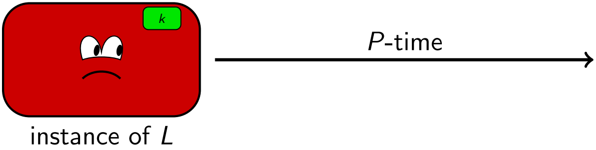

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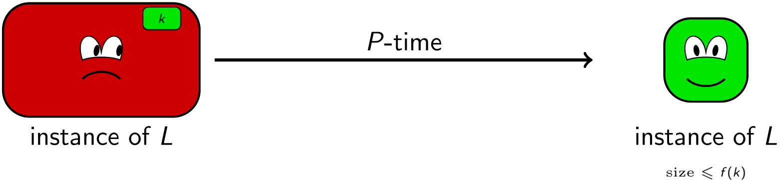

Background in complexity theory

Unparameterized problems

$\Leftrightarrow$

Languages over Σ, for a finite alphabet Σ

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- **Unparameterized variant**: k is appended to x in unary.
- **Kernelization algorithm** takes on input an instance (x, k), and outputs an instance (x', k') such that

$$(x, k) \in L \Leftrightarrow (x', k') \in L \qquad \text{and} \qquad |x'| + k' \leqslant f(k)$$

for some computable function f.

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- Question of existence of **any kernel** is equivalent to being FPT.
- We are interested in **polynomial kernels**, where f is a polynomial.
- Before 2008, no tool to classify FPT problems wrt. whether they have polykernels or not.

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Intuition

The final number of bits is much less than the number input instances. Most of the instances have to be **discarded completely**.

Kernelization and Compression

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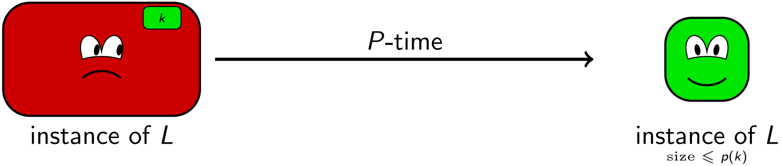

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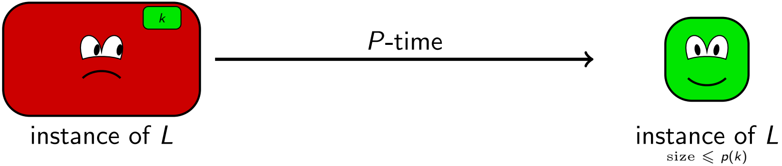

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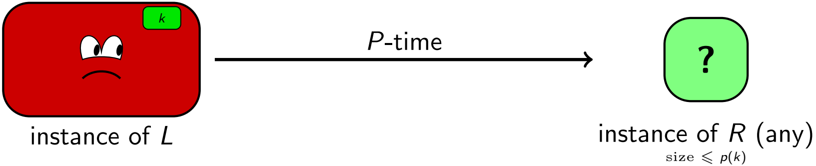

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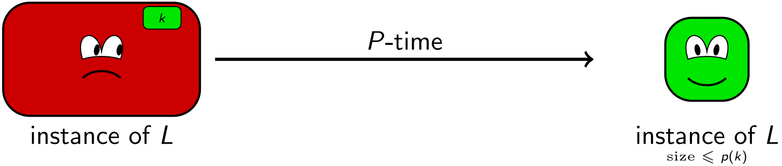

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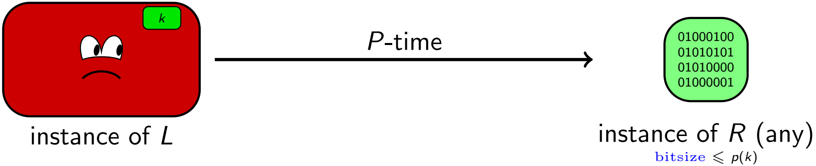

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- A polynomial compression can be turned into a polynomial kernelization provided that there is a **P**-reduction from R to L.
  - For instance, when $R \in \mathbf{NP}$ and L is **NP**-hard.
- **Note**: There are examples when a poly-compression is known but a poly-kernel is not known, because it is unclear whether R is in **NP**.

OR-distillation

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OR-distillation of L into R

Input: Words $x_1, x_2, \ldots, x_t$, each of length at most k.
Time: $\mathrm{poly}(t + \sum_{i=1}^{t} |x_i|)$.
Output: One word y such that
(a) $|y| = \mathrm{poly}(k)$, and
(b) $y \in R$ if and only if $x_i \in L$ for at least one i.

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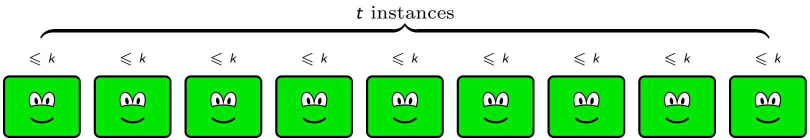

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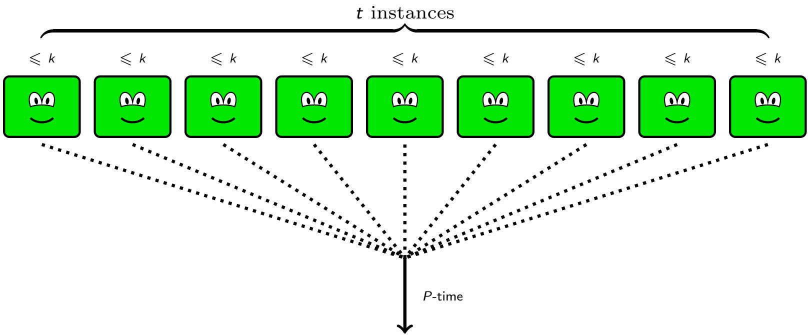

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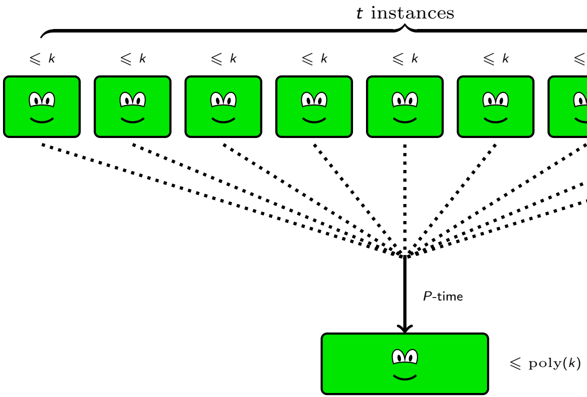

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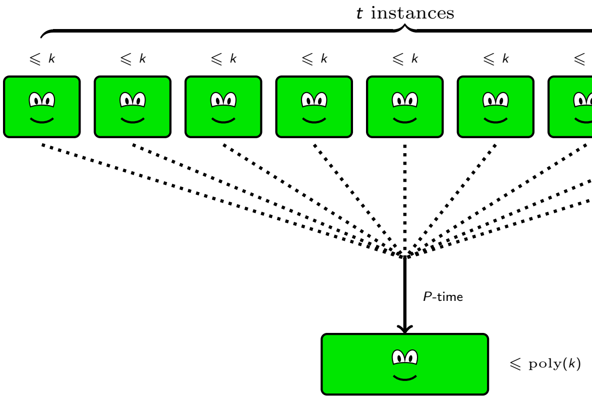

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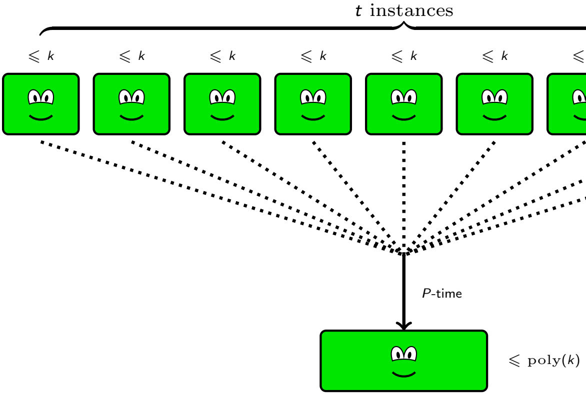

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Define $\text{OR-}L = \{x_1 \# x_2 \# \ldots \# x_t \ : \ x_i \in L \text{ for at least one } i\}$.

OR-distillation $L \rightarrow R$ is a polynomial compression OR-$L/\max|x_i| \rightarrow R$

Backbone theorem

OR-distillation theorem [Fortnow, Santhanam; 2008]

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  - Not as bad as **P** $=$ **NP**, but still considered very unlikely.
- The proof is very short, but very tricky.

OR-composition

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OR-composition algorithm for L

Input: Instances $(x_1, k), (x_2, k), \ldots, (x_t, k)$.
Time: $\mathrm{poly}(t + \sum_{i=1}^{t} |x_i| + k)$.
Output: One instance $(y, k^\star)$ such that
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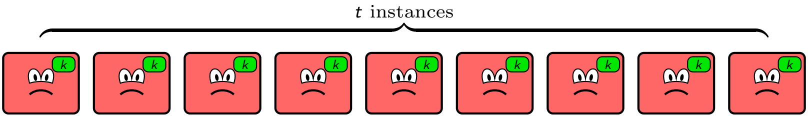

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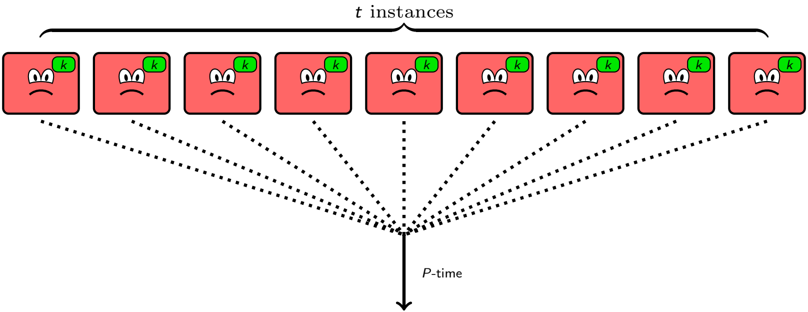

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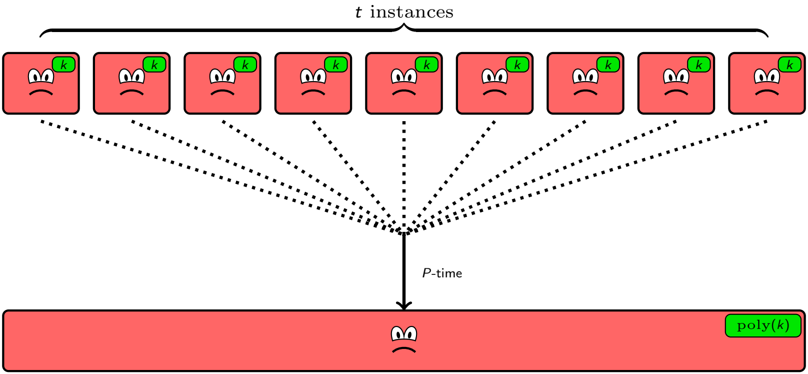

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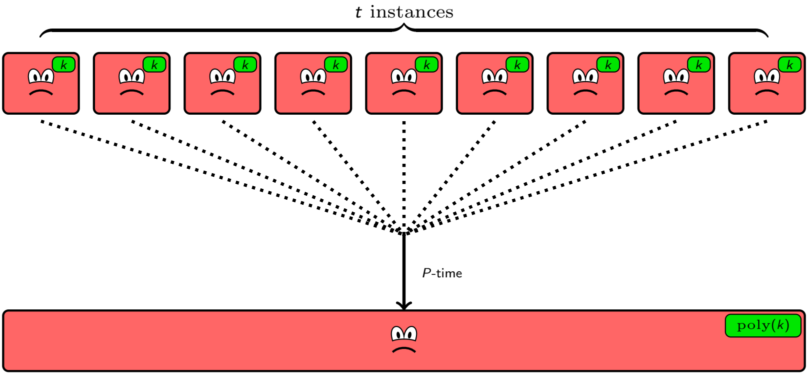

OR-composition theorem [BDFH; 2008]

Suppose a parameterized problem L admits an OR-composition algorithm, and the unparameterized version of L is **NP**-hard.

Then L does not admit a polynomial kernel unless **NP** $\subseteq$ **coNP**/poly.

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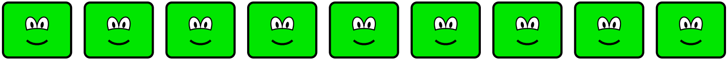

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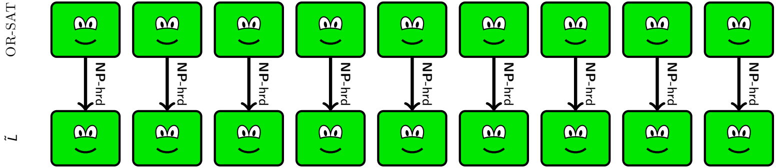

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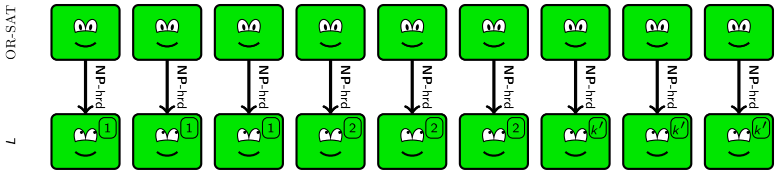

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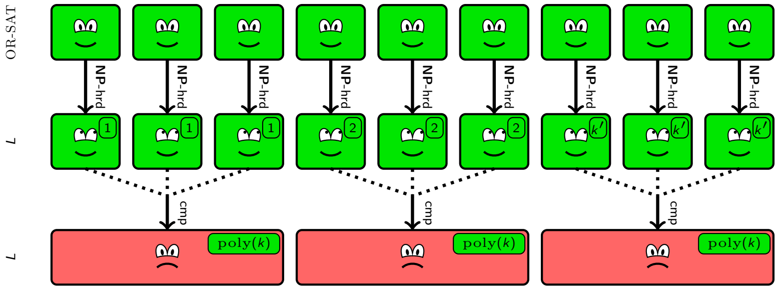

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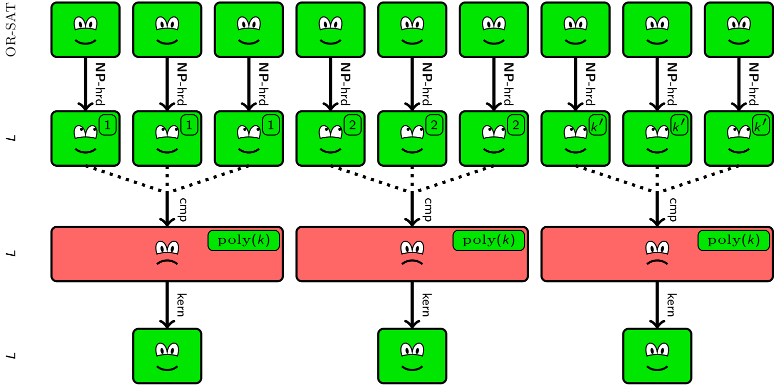

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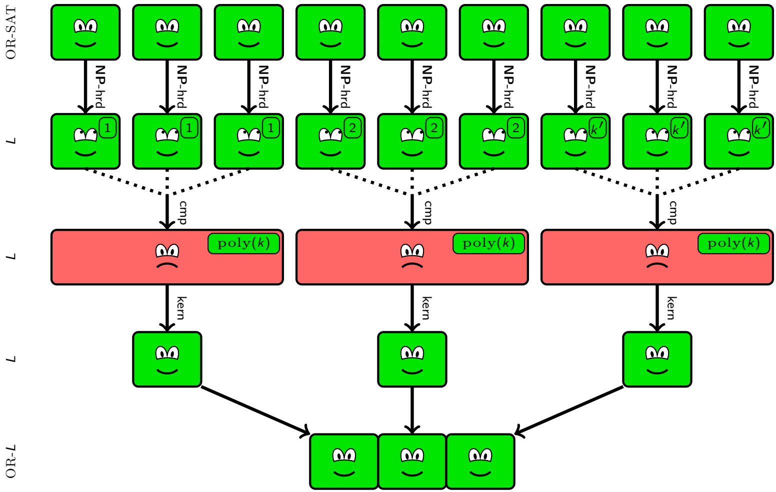

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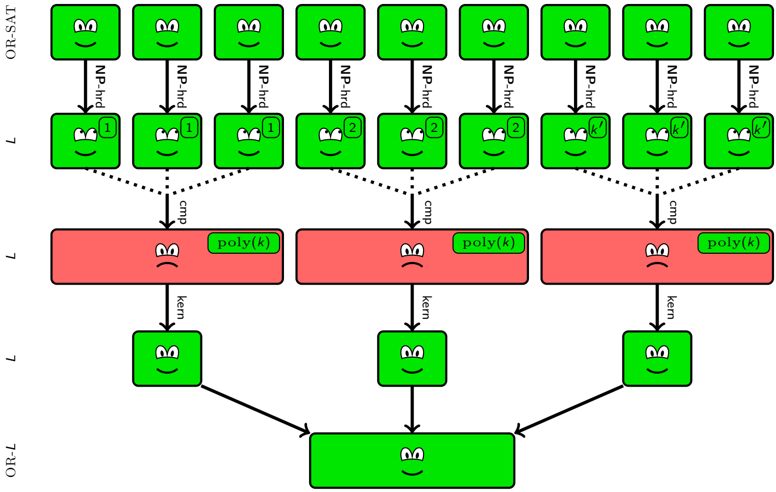

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- Same for k-Cycle and many other problems.
- Today, investigating the existence of a polynomial kernel is often a secondary goal after showing that a problem is FPT.

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  - Yes, as long as we have polynomial number of buckets.
- How large can t be?
  - Well, not larger than $|\Sigma|^{k+1}$, as we may remove duplicates of the input instances.
  - Hence, we may assume that $\log t = \mathcal{O}(k)$.

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  - Observed also earlier via different arguments. (Dom, Lokshtanov, and Saurabh; 2009)

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  - **Examples**: Max Leaf Subtree, Set Cover/m, Set Cover/n, Steiner Tree, Connected Vertex Cover, Disjoint Paths, Directed Multiway Cut with 2 terminals, ...
- Most of the works use a subset of mentioned features.
- **Later**: a new formalism **cross-composition** gathers all the features. (Bodlaender, Jansen, and Kratsch; 2011)

Polynomial equivalence relation

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Equivalence relation $\sim$ on $\Sigma^\star$ is a **polynomial equivalence relation** if:

- checking whether two words $x, y \in \Sigma^\star$ are $\sim$-equivalent can be done in $\mathrm{poly}(|x| + |y|)$ time; and
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An unparameterized problem Q **cross-composes** into a parameterized problem L, if there exists a polynomial equivalence relation $\sim$ and an algorithm that, given $\sim$-equivalent strings $x_1, x_2, \ldots, x_t$, in time $\mathrm{poly}\left(t + \sum_{i=1}^{t} |x_i|\right)$ produces one instance $(y, k^\star)$ such that

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Cross-composition theorem [Bodlaender, Jansen, Kratsch]

If some **NP**-hard problem Q cross-composes into L, then L has no polynomial compression into any language R, unless **NP** $\subseteq$ **coNP**/poly.

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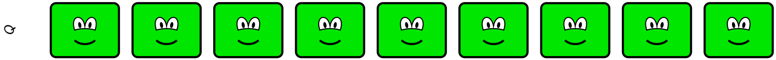
$k = \max |x_i|, \quad \log t = \mathcal{O}(k)$



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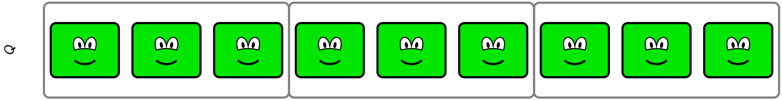

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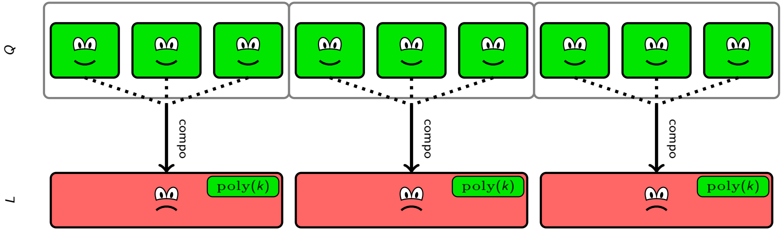

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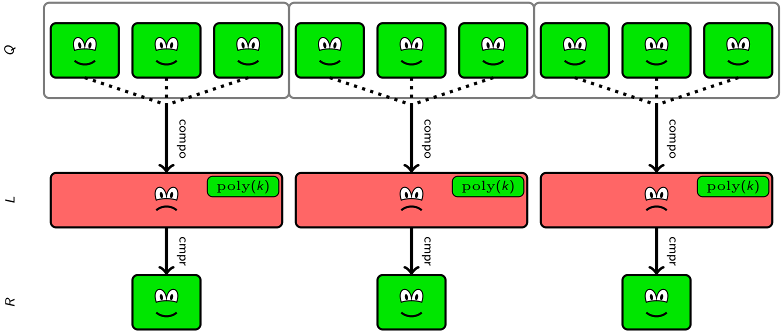

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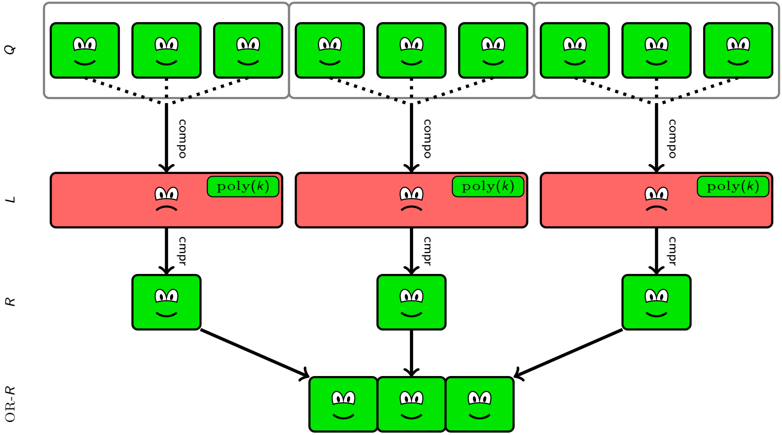

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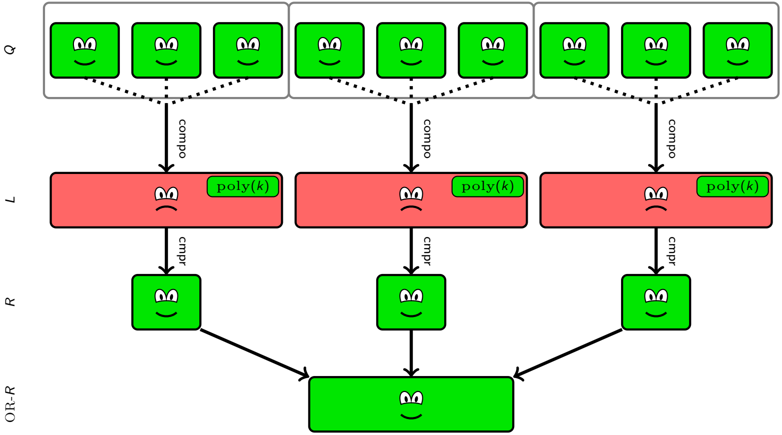

Applications

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- In fact, cross-composition is a good framework to express also all the previous results.
- **Plan for now**: show some non-trivial cross-composition to give an intuition about basic tricks.

Application 1: Set Splitting

Set Splitting

I: Universe U and family of subsets $\mathcal{F} \subseteq 2^U$

P: $|U|$

Q: Is there a coloring $\mathcal{C} : U \to \{\mathbf{B}, \mathbf{W}\}$ such that every set $X \in \mathcal{F}$ is split, i.e., contains a black and a white element?

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- Assume that t is a power of 2 (by copying the instances).

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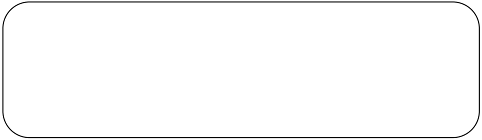

PLAYGROUND

joint universe U

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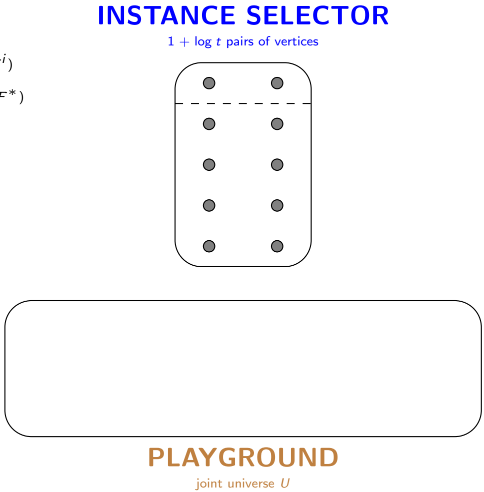

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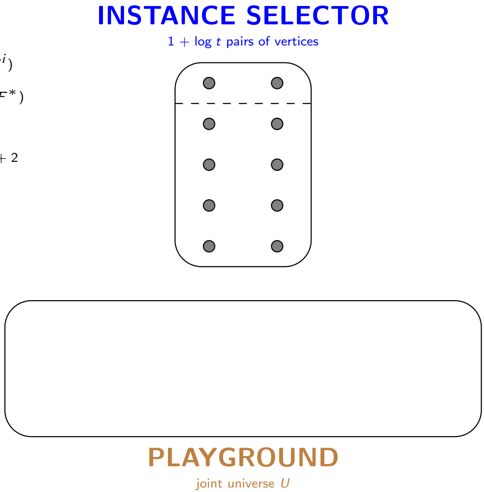

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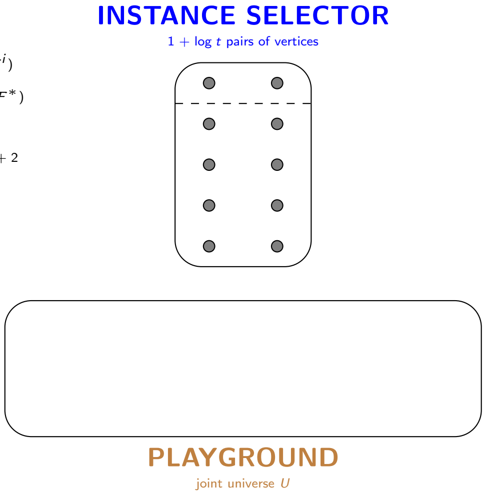

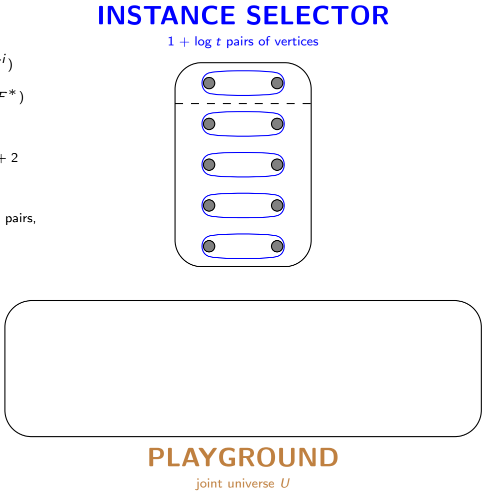

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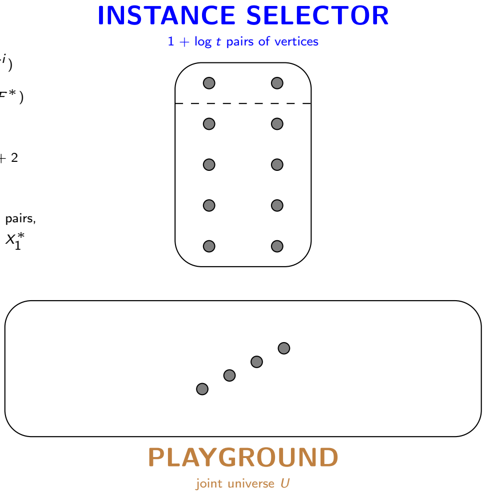

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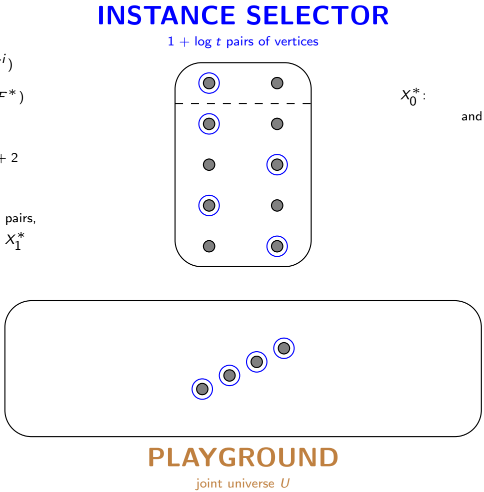

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Cross-composing into SET SPLITTING

INSTANCE SELECTOR

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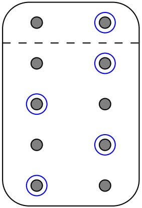

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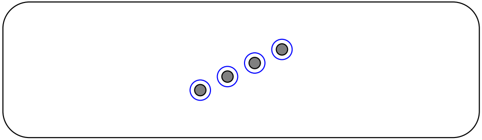

PLAYGROUND

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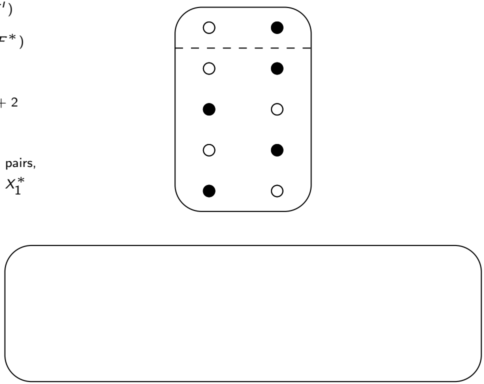

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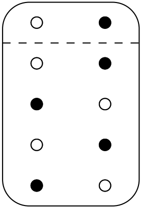

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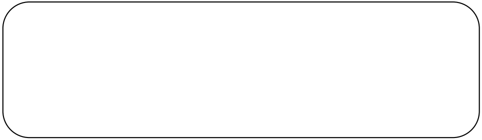

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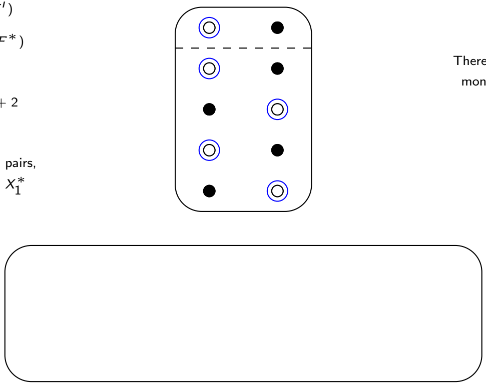

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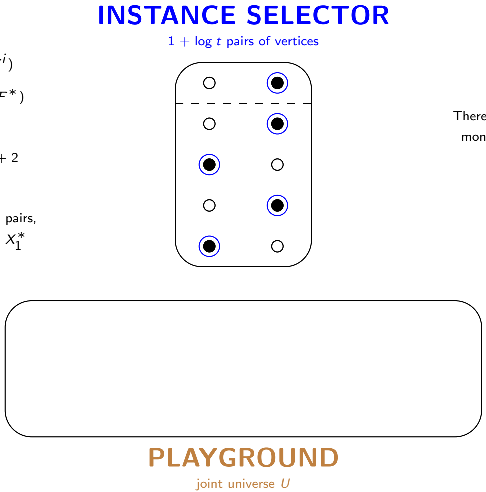

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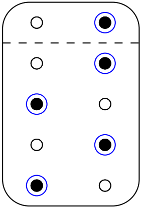

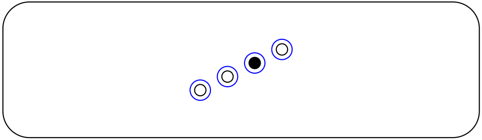

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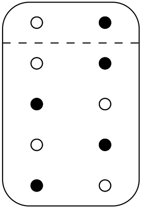

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($\Leftarrow$): If $(U, \mathcal{F}^i)$ is solvable, we set IS accordingly, and solve this instance in PL. Remaining sets are split for free.

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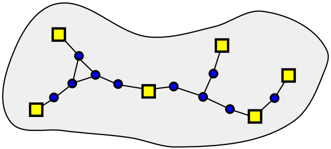
- **Proof**:
  Compose the PPT with the assumed compression for Q.

Application 2: Steiner Tree

Steiner Tree

I: Graph G with terminals $T \subseteq V(G)$, $k \in \mathbb{N}$
P: $k + |T|$
Q: Is there a set $X \subseteq V(G) \setminus T$, such that $|X| \leqslant k$ and $G[T \cup X]$ is connected?



Application 2: Steiner Tree

Steiner Tree

I: Graph G with terminals $T \subseteq V(G)$, $k \in \mathbb{N}$

P: $k + |T|$

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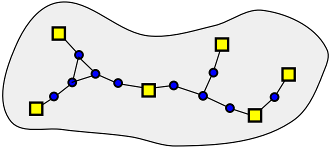
- We show that Steiner Tree has no polykernel (unless...) using a PPT from a auxiliary problem.



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- **High level**: Extract the essence of the original problem into the auxiliary problem.

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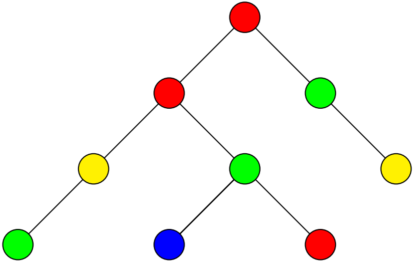

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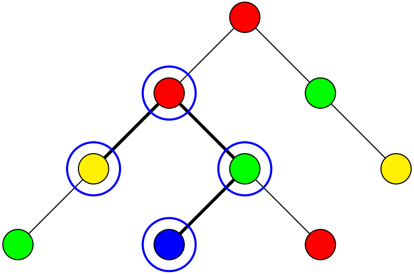

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- **Now**: PPT from CGM to ST.

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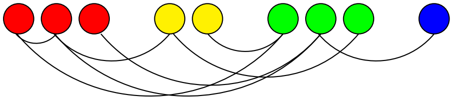

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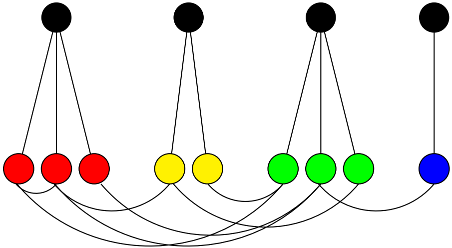

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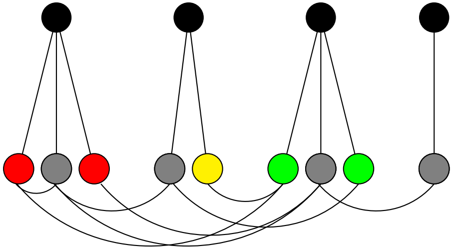

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- **Note**: Composition for CGM is far simpler than trying to do this directly for Steiner Tree.

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- **Corollary**: The whole framework works for AND instead of OR.

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- **Note**: The $2k$-kernel for VC needs $\mathcal{O}(k^2)$ bits for the encoding.

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- **Thank you for your attention!**