ETH and SETH lower bounds

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- For every q you can do $(2-\varepsilon_q)^n$ for $q\mathrm{SAT}$, but $\lim_{q\to\infty} \varepsilon_q = 0$.

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ETH and SETH, first attempt

ETH: $\mathrm{3SAT}$ cannot be solved in time $2^{o(n)}$.
SETH: $\mathrm{CNF\text{-}SAT}$ cannot be solved in time $\mathcal{O}(\alpha^n)$ for any $\alpha < 2$.

Actual statements

$$\delta_q = \inf\{c : \text{There is an } \mathcal{O}(2^{cn}) \text{ algorithm for } q\text{SAT}\}$$

Exponential Time Hypothesis, ETH

$$\delta_3 > 0.$$

There is a $c > 0$ such that $3\mathrm{SAT}$ cannot be solved in $\mathcal{O}(2^{cn})$ time.

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- ETH is widely believed, SETH is disputed.

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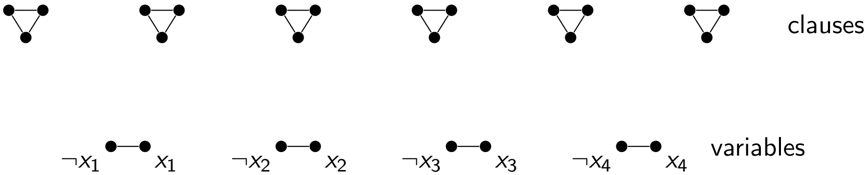
Vertex Cover

I: graph G and $k \in \mathbb{N}$
Q: Is there a set $X \subseteq V(G)$ with $|X| \leqslant k$ such that every edge of G has at least one endpoint in X?

Let us inspect the standard reduction from 3SAT to Vertex Cover.



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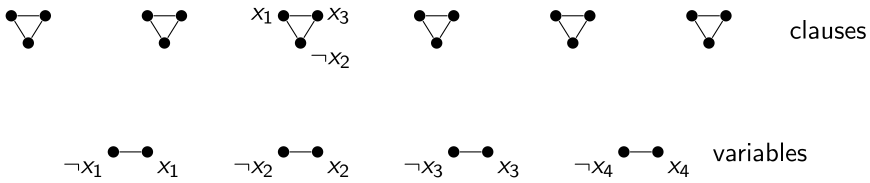

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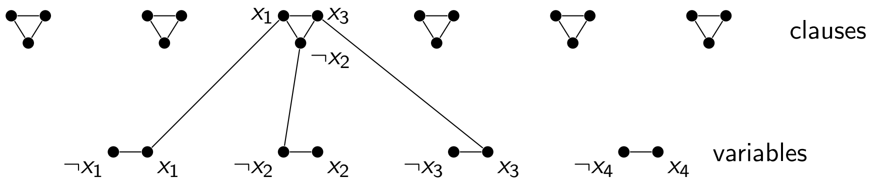

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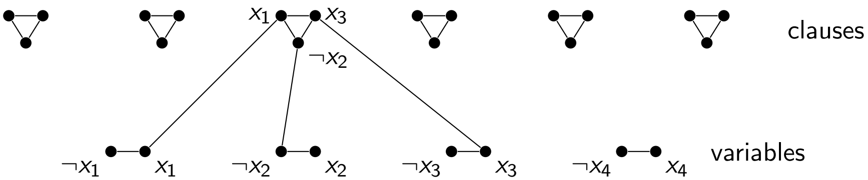

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φ is satisfiable iff the created graph has a vertex cover of size $n + 2m$.

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- **Gap**: between upper bound $2^{\mathcal{O}(N)}$ and $2^{o(N^{1/3})}$.
- If we started with an instance of 3-SAT that was **sparse**, that is, $m = \mathcal{O}(n)$, then a $2^{o(N)}$ lower bound would follow.

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Sparsification Lemma [IPZ, 2001]

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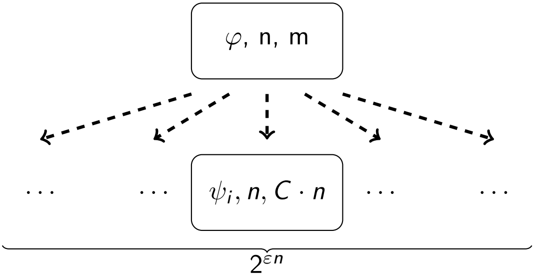

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- Use Sparsification Lemma for $\varepsilon = d/2$. Denote $C = C(3, \varepsilon)$.
- Solve each ψ_i by $\mathcal{A}_{c'}$, where $c' = \frac{d}{2(C+1)}$.
- The total running time is

$$\mathcal{O}^\star(2^{\varepsilon n}) + \mathcal{O}^\star(2^{\varepsilon n} \cdot 2^{\frac{d}{2(C+1)} \cdot (C+1)n}) = \mathcal{O}^\star(2^{dn}).$$

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- Consider any $d > 0$. We want to solve $3\mathrm{SAT}$ in time $\mathcal{O}^\star(2^{dn})$.
- Use Sparsification Lemma for $\varepsilon = d/2$. Denote $C = C(3, \varepsilon)$.
- Solve each ψ_i by $\mathcal{A}_{c'}$, where $c' = \frac{d}{2(C+1)}$.
- The total running time is

$$\mathcal{O}^\star(2^{\varepsilon n}) + \mathcal{O}^\star(2^{\varepsilon n} \cdot 2^{\frac{d}{2(C+1)} \cdot (C+1)n}) = \mathcal{O}^\star(2^{dn}).$$

Corollary

Under ETH, there is no $2^{o(n+m)}$-time algorithm for $3\mathrm{SAT}$, so also no $2^{o(N+M)}$-time algorithm for Vertex Cover.

Transfer of hardness

Hardness under ETH via reductions

Suppose there is a reduction from $3\mathrm{SAT}$ to a problem L that produces an instance of size $N \leqslant f(n+m)$.

Then we can exclude an algorithm for L with running time $2^{o(f^{-1}(N))}$.

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- Let's see tight bounds for another form of running time.

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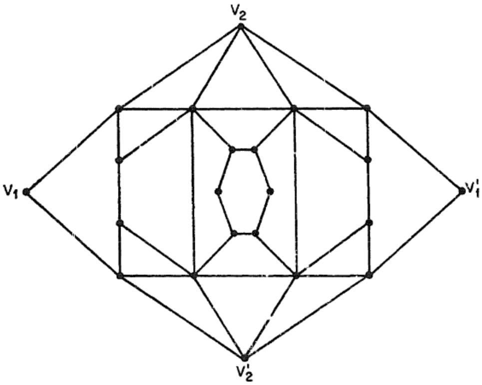

Fig. 11. Crossover H for Theorem 2.7.

PLVC: analysis

- Reduction takes an instance of VC with n vertices and m edges, and outputs an instance with $N = \mathcal{O}((n+m)^2)$ vertices.

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- The $\sqrt{N}$ term in the exponent is not a coincidence!

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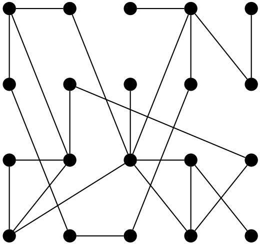
- We start from an instance of 3-COLORING, for which we already have a $2^{o(n)}$ lower bound.

ETH hardness for CLIQUE



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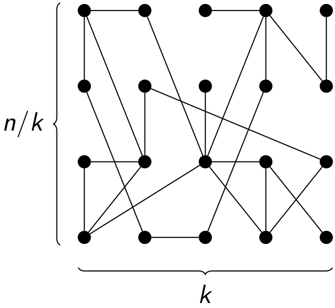

Partition vertices into k groups with n/k vertices each.

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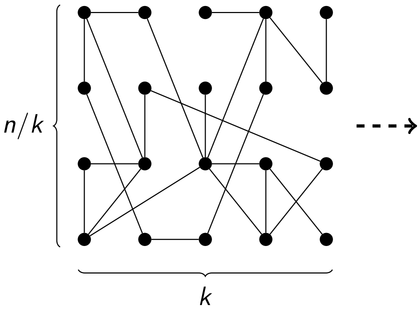

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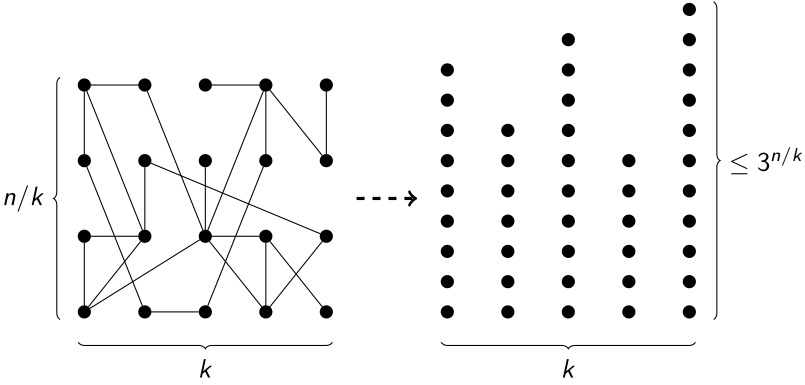

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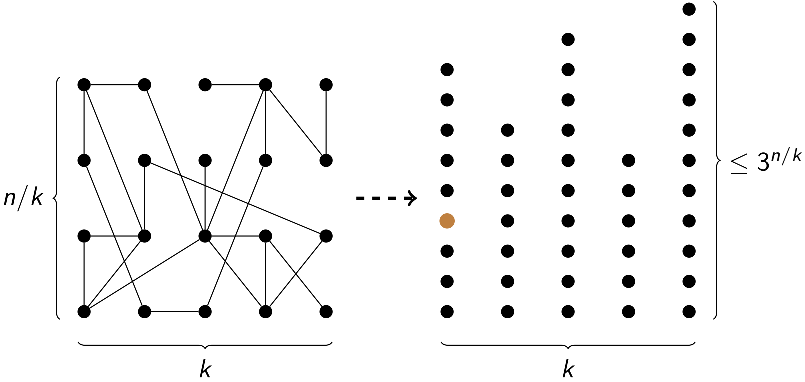

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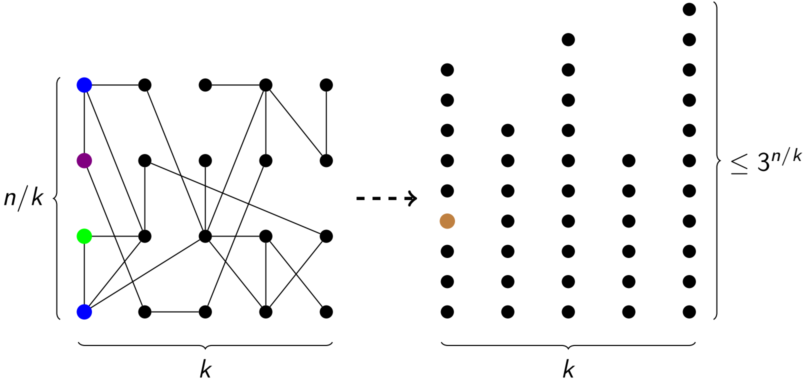

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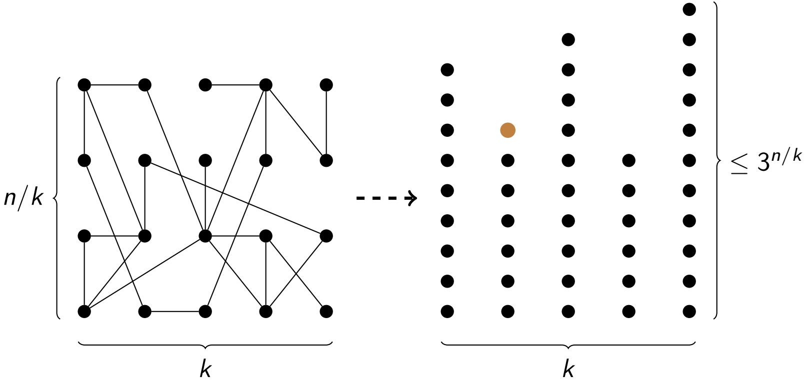

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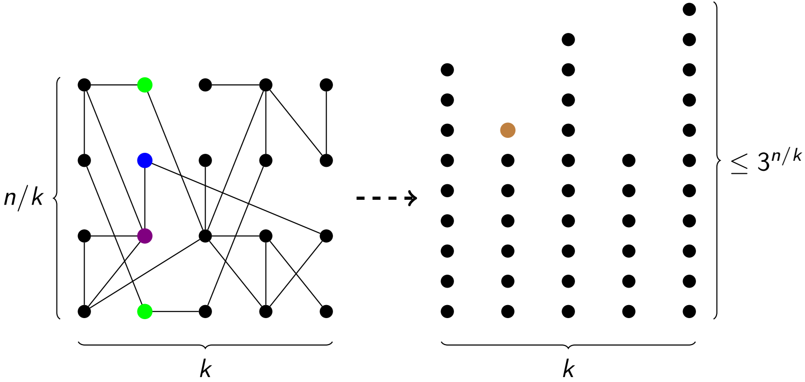

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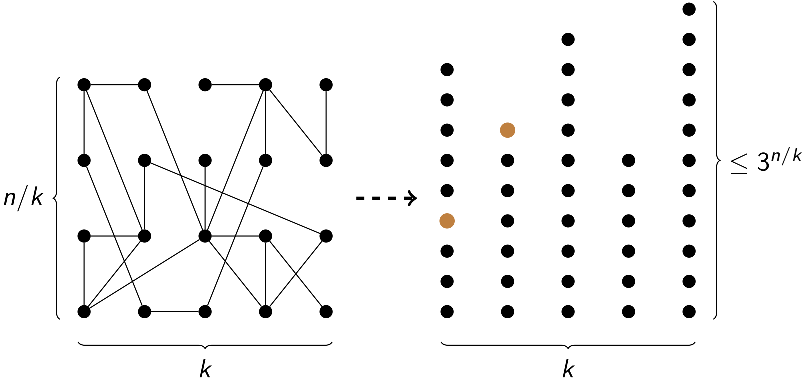

Connect two colorings if they are consistent.

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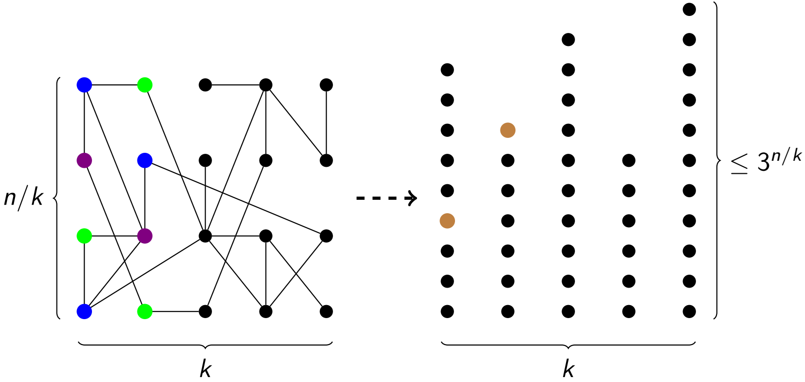

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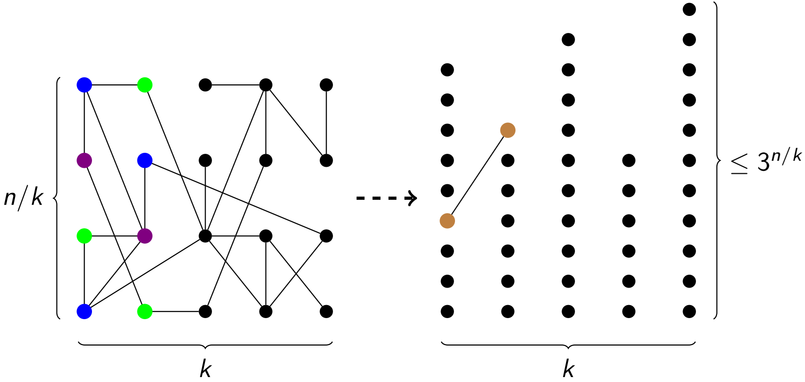

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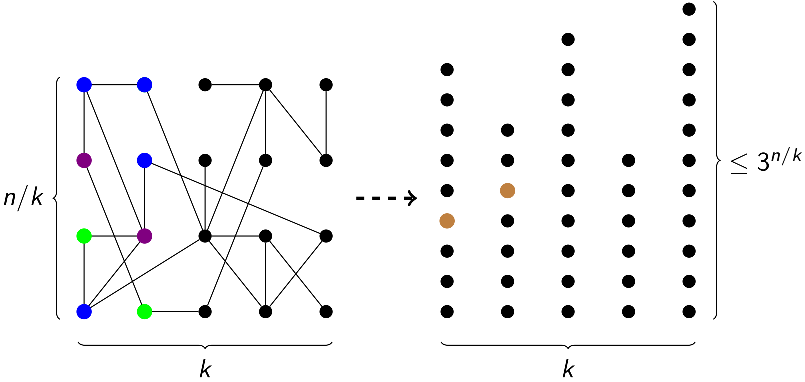

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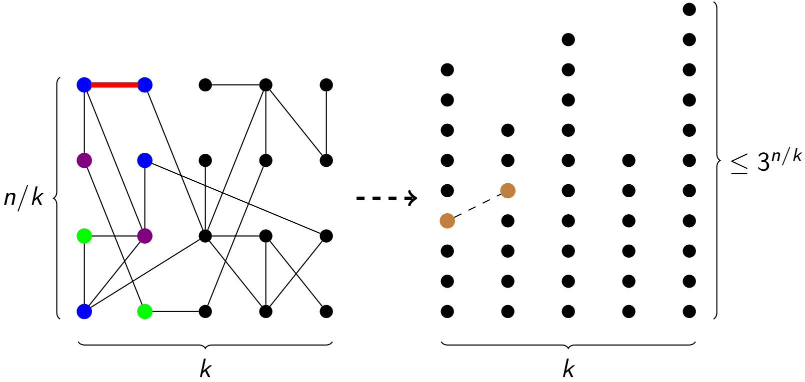

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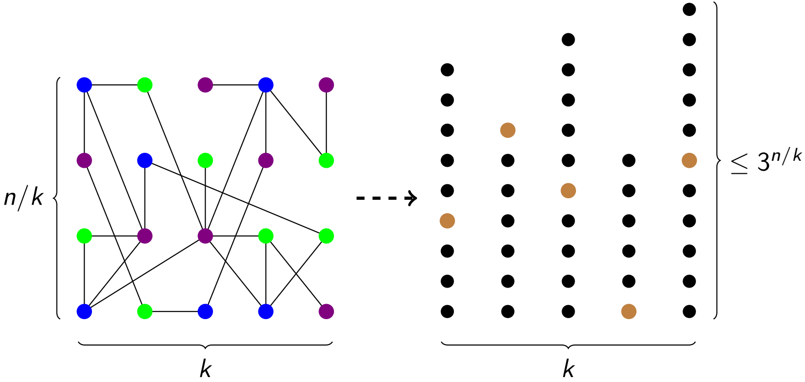

A 3-coloring on the left corresponds to a k-clique on the right.

Left graph admits 3-coloring iff right graph contains k-clique.

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- **Intuition**: We embed the 3^n solution space of 3-COLORING into the N^k solution space of CLIQUE.

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  - The reduction transforms an instance of CLIQUE with parameter k into an instance of PLANAR SCATTERED SET with parameter $\mathcal{O}(k^2)$.

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Lower bounds for parameterized problems under ETH

Suppose L admits a polynomial-time reduction from $3\mathrm{SAT}$ with output parameter $k \leqslant f(n+m)$.

Then L does not admit an $\mathcal{O}^\star(2^{o(f^{-1}(k))})$ time algorithm unless ETH fails.

ETH and parameterized complexity

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- We focus on (b), since this is the most typical behavior in parameterized algorithms.

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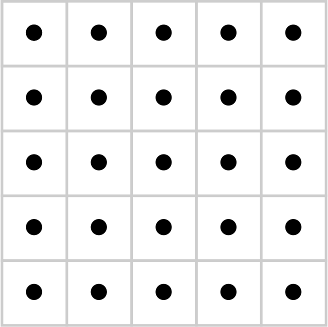

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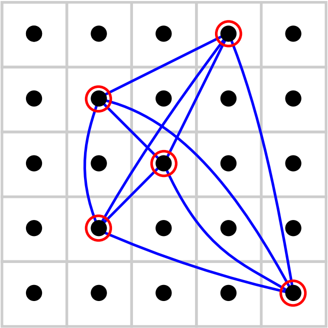

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Corollary

Unless ETH fails, there is no algorithm for $k \times k$-Clique with running time $\mathcal{O}^\star(2^{o(k \log k)})$.

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  - **Note**: For Hamiltonian Path you can get running time $\mathcal{O}^\star(4^t)$.

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  - Also, technically more challenging.

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- **Open**: The algorithmic result **does not** work for treewidth.

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SETH and covering problems

Set Cover

I: Universe U, set family $\mathcal{F} \subseteq 2^U$, integer k
Q: Is there a subfamily $\mathcal{G} \subseteq \mathcal{F}$ with $|\mathcal{G}| \leqslant k$ s.t. $\bigcup \mathcal{G} = U$?

- Denote $n = |U|$ and $m = |\mathcal{F}|$.
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Set Cover Conjecture

Let λ_q be the infimum of the set of constants c such that q-Set Cover can be solved in time $\mathcal{O}^\star(2^{cn})$, where n is the size of the universe. Then $\lim_{q\to\infty} \lambda_q = 1$. In particular, there is no algorithm for the general Set Cover problem that runs in $\mathcal{O}^\star(\alpha^n)$ for any $\alpha < 2$.

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- **Example**: Longest Common Subsequence in strongly subquadratic time.

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- **Note**: dimension is $m \leqslant Cn \leqslant 2C \log N$.

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- **Thank you for your attention!**