FPT in Economics

The Stable Marriage Problem

Meirav Zehavi
Outline

- Introduction
- Basics
- Satisfaction Optimization
- Size Optimization
- Minimum BPs, Manipulation
- Roommate, Hospitals/Residents
Input. A set of men $M$, and a set of women $W$. 
**Input.** A set of men $M$, and a set of women $W$. Every agent has a set of acceptable partners.
Input. A set of men $M$, and a set of women $W$. Every agent has a set of acceptable partners. The acceptable partners are ranked. Bijective function $p_m : W' \rightarrow \{1, \ldots, |W'|\}$. 
Input.
- Complete/incomplete lists.
Input.
- Complete/incomplete lists.
- Ties allowed/forbidden.

Surjective function $\rho_m : W' \rightarrow \{1, \ldots, t\}$, $t \leq |W'|$. 
Matching. A set of pairwise-disjoint pairs, each consisting of a man and a woman that find each other acceptable.
Blocking pair. A pair \((m, w)\) blocks a matching if \(m\) and \(w\) prefer being matched to each other to their current ```status```.
Blocking pair. A pair \((m,w)\) blocks a matching if \(m\) and \(w\) prefer being matched to each other to their current ``status''.

**Blocking pair**
Stable matching. A matching that has no blocking pair.
Stable Marriage problem. Find a stable matching.
Stable Marriage problem. Find a stable matching.

Nobel Prize in Economics, 2012. Awarded to Shapley and Roth “for the theory of stable allocations and the practice of market design.”
Introduction

Applications.
- Matching hospitals to residents.
- Matching students to colleges.
- Matching kidney patients to donors.
- Matching users to servers in a distributed Internet service.
Introduction

Books.

Surveys.
Iwama and Miyazaki, 2008; Gupta, Roy, Saurabh and Zehavi, 2017.
**Primal graph.** A bipartite graph with bipartition \((M,W)\), where \(m\) and \(w\) are adjacent iff they find each other acceptable.
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Proposition. A stable matching can be found in time $O(n^2)$. [Gale and Shapley, 1962]

→ A stable matching always exists.
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There can be an exponential number of stable matchings. [Gusfield and Irving, 1989]

Proposition. All stable matchings match the same set of agents. [Gale and Sotomayor, 1985]
A “spectrum” of stable matchings, where the two extremes are the **man-optimal** stable matching and the **woman-optimal** stable matching.
A "spectrum" of stable matchings, where the two extremes are the man-optimal stable matching and the woman-optimal stable matching.

Man-optimal stable matching $\mu_M$. For every stable matching $\mu$ and man $m$, either $m$ is unmatched by both $\mu_M$ and $\mu$, or $p_m(\mu_M(m)) \leq p_m(\mu(m))$. 
A "spectrum" of stable matchings, where the two extremes are the man-optimal stable matching and the woman-optimal stable matching.

**Man-optimal stable matching** $\mu_M$. For every stable matching $\mu$ and man $m$, either $m$ is unmatched by both $\mu_M$ and $\mu$, or $p_m(\mu_M(m)) \leq p_m(\mu(m))$.

→ Unique.
A ``spectrum'' of stable matchings, where the two extremes are the **man-optimal stable matching** and the **woman-optimal stable matching**.

**Man-optimal stable matching** $\mu_M$. For every stable matching $\mu$ and man $m$, either $m$ is unmatched by both $\mu_M$ and $\mu$, or $p_m(\mu_M(m)) \leq p_m(\mu(m))$.

**Proposition.** $\mu_M$ and $\mu_W$ exist, and can be found in time $O(n^2)$. [Gale and Shapley, 1962]
Rotation. A $\mu$-rotation is an ordered sequence $\rho = ((m_0, w_0), (m_1, w_1), ..., (m_{r-1}, w_{r-1}))$ such that for all $i$,

- $(m_i, w_i) \in \mu$, and
- $w_{(i+1) \mod r}$ is the woman succeeding $w_i$ in $m_i$’s preference list who prefers being matched to $m_i$ to her current status.

$$m_i: \quad \ldots \quad w_i \quad \ldots \quad w_{(i+1) \mod r} \quad \ldots$$
Rotation. A \( \mu \)-rotation is an ordered sequence \( \rho = ((m_0, w_0), (m_1, w_1), \ldots, (m_{r-1}, w_{r-1})) \) such that for all \( i \),

- \((m_i, w_i) \in \mu\), and
- \(w_{(i+1) \text{mod} \ r}\) is the woman succeeding \( w_i \) in \( m_i \)'s preference list who prefers being matched to \( m_i \) to her current status.

\( \rho \) is a rotation if it is a \( \mu \)-rotation for some \( \mu \).
**Rotation.** A $\mu$-rotation is an ordered sequence $\rho = ((m_0,w_0),(m_1,w_1),..., (m_{r-1},w_{r-1}))$ such that for all $i$,

- $(m_i,w_i) \in \mu$, and
- $w_{(i+1)\mod r}$ is the woman succeeding $w_i$ in $m_i$’s preference list who prefers being matched to $m_i$ to her current status.

$\rho$ is a rotation if it is a $\mu$-rotation for some $\mu$.

The set of all rotations is denote by $R$. It is known that $|R| \leq n^2$. 
Rotation elimination. Consider a $\mu$-rotation $\rho = ((m_0,w_0),(m_1,w_1),..., (m_{r-1},w_{r-1}))$. The elimination of $w_i$ is the operation that modifies $\mu$ by matching each $m_i$ with $w_{(i+1) \mod r}$ rather than $w_i$. 
Rotation elimination. Consider a $\mu$-rotation $\rho = ((m_0, w_0), (m_1, w_1), ..., (m_{r-1}, w_{r-1}))$. The elimination of $\mu$ is the operation that modifies $\mu$ by matching each $m_i$ with $w_{(i+1) \mod r}$ rather than $w_i$. 
Rotation elimination. Consider a $\mu$-rotation $\rho = ((m_0, w_0), (m_1, w_1), ..., (m_{r-1}, w_{r-1}))$.
The elimination of is the operation that modifies $\mu$ by matching each $m_i$ with $w_{(i+1) \mod r}$ rather than $w_i$.

Rotation elimination results in a stable matching. [Irving and Leather, 1986]
Proposition. Let $\mu$ be a stable matching. There is a unique subset of $R$, denoted by $R(\mu)$, such that starting from $\mu_M$, there is an order in which the rotations in $R(\mu)$ can be eliminated to obtain $\mu$. [Irving and Leather, 1986]
Rotation poset. $\mathcal{P}=(R,\prec)$, where $\prec$ is a partial order on $R$ such that $\rho \prec \rho'$ iff for every stable matching $\mu$, if $\rho'$ is in $R(\mu)$, then $\rho$ is also in $R(\mu)$. 
Rotation poset. $\mathcal{P}=(R,<)$, where $<$ is a partial order on $R$ such that $\rho < \rho'$ iff for every stable matching $\mu$, if $\rho'$ is in $R(\mu)$, then $\rho$ is also in $R(\mu)$.

- Elimination compatible with $<$. 
Rotation poset. $\mathcal{P}=(R,\prec)$, where $\prec$ is a partial order on $R$ such that $\rho \prec \rho'$ iff for every stable matching $\mu$, if $\rho'$ is in $R(\mu)$, then $\rho$ is also in $R(\mu)$.

- Elimination compatible with $\prec$.
- Closed set $R'$. If $\rho \in R'$, then $\rho' \in R'$ for all $\rho' \prec \rho$. 
Proposition. Let $R'$ be a closed set. Starting with $\mu_M$, eliminating the rotations in $R'$ in any $\prec$-compatible order is valid—at each step, where the current stable matching is $\mu$, the rotation we eliminate next is a $\mu$-rotation.
**Proposition.** Let $R'$ be a closed set. Starting with $\mu_M$, eliminating the rotations in $R'$ in any $\prec$-compatible order is valid—at each step, where the current stable matching is $\mu$, the rotation we eliminate next is a $\mu$-rotation. Moreover, all $\prec$-compatible orders in which one eliminates the rotations in $R'$ result in the same stable matching. [Irving and Leather, 1986]
Rotation digraph. A compact representation of $\prod$. The rotation digraph is the DAG of minimum size whose transitive closure is isomorphic to $\prod$. 
Rotation digraph. A compact representation of $\Pi$. The rotation digraph is the DAG of minimum size whose transitive closure is isomorphic to $\Pi$. 

$\Pi$ (partial)
**Rotation digraph.** A compact representation of $\prod$. The rotation digraph is the DAG of minimum size whose transitive closure is isomorphic to $\prod$.

**Proposition.** The rotation digraph can be computed in time $O(n^2)$. [Irving, Leather and Gusfield, 1987]
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Recall.

man-optimal

woman-optimal
Recall. man-optimal woman-optimal

Three satisfaction optimization approaches.
• Globally desirable.
• Fair towards both sides.
• Desirable by both sides.
Recall.

- man-optimal
- woman-optimal

Three satisfaction optimization approaches.

- Globally desirable.
- Fair towards both sides.
- Desirable by both sides.

No ties.
Egalitarian stable matching. Minimize

$$e(\mu) = \sum_{(m,w) \in \mu} (pm(w) + pw(m)).$$
Egalitarian stable matching. Minimize

\[ e(\mu) = \sum_{(m,w) \in \mu} \left( pm(w) + pw(m) \right) . \]

**Comment.** In the presence of ties, can use either the definition above or one where each unmatched agent \( a \) contributes \( |\text{domain}(p_a)| + 1 \) to the sum.
Proposition. Egalitarian Stable Marriage is solvable in polynomial time. [Irving, Leather and Gusfield, 1987]
Proposition. Egalitarian Stable Marriage is solvable in polynomial time. [Irving, Leather and Gusfield, 1987]

In the presence of ties (NP-hard):
Proposition. Egalitarian Stable Marriage is FPT parameterized by "total length of ties". [Marx and Schlotter, 2010]
Sex-equality measure.

\[ \delta(\mu) = \sum_{(m,w) \in \mu} p_m(w) - \sum_{(m,w) \in \mu} p_w(m). \]
Sex-equality measure.

\[ \delta(\mu) = \sum_{(m,w) \in \mu} p_m(w) - \sum_{(m,w) \in \mu} p_w(m). \]

Sex-Equal Stable Marriage. Find a stable matching that attains \( \Delta = \min_{\mu} |\delta(\mu)|. \)
Sex-equality measure.

\[ \delta(\mu) = \sum_{(m,w) \in \mu} p_m(w) - \sum_{(m,w) \in \mu} p_w(m). \]

Sex-Equal Stable Marriage. Find a stable matching that attains \( \Delta = \min_{\mu} |\delta(\mu)|. \)

Proposition. Sex-Equal Stable Marriage is NP-hard. [Kato, 1993]
**Propositions.** [Gupta, Saurabh and Zehavi, 2017]

1. Sex-Equal Stable Marriage is W[1]-hard w.r.t $tw$, the treewidth of the primal graph. Moreover, unless ETH fails, it cannot be solved in time $f(tw)n^{o(tw)}$. 
Propositions. [Gupta, Saurabh and Zehavi, 2017]

1. Sex-Equal Stable Marriage is W[1]-hard w.r.t $tw$, the treewidth of the primal graph. Moreover, unless ETH fails, it cannot be solved in time $f(tw) n^{o(tw)}$.

2. Sex-Equal Stable Marriage can be solved in time $n^{O(tw)}$. 
Propositions. [Gupta, Saurabh and Zehavi, 2017]
1. Sex-Equal Stable Marriage is solvable in time $2^{tw}n^{O(1)}$, where $tw$ is the treewidth of the rotation digraph.
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1. Sex-Equal Stable Marriage is solvable in time $2^{tw}n^{O(1)}$, where $tw$ is the treewidth of the rotation digraph.

2. Unless SETH fails, Sex-Equal Stable Marriage cannot be solved in time $(2-\varepsilon)^{tw}n^{O(1)}$. 
**Proposition.** Sex-Equal Stable Marriage is solvable in time \((2^{\alpha n} + 2^\beta)^tw n^{O(1)}\) for \(\alpha = (5 - \sqrt{24}) (t-2+\epsilon)\) and \(\beta = (t-1)/2\epsilon\), where \(t\) is the maximum length of a list.

[McDermid and Irving, 2014]
Balance measure. $bal(\mu) = \max\{ \sum_{(m,w) \in \mu} p_m(w), \sum_{(m,w) \in \mu} p_w(m) \}$. 
Balance measure. \[ bal(\mu) = \max \left\{ \sum_{(m,w) \in \mu} p_m(w), \sum_{(m,w) \in \mu} p_w(m) \right\}. \]

Balanced Stable Marriage. Find a stable matching that attains \( \text{Bal} = \min_\mu bal(\mu) \).
Balance measure. \[ bal(\mu) = \max\left\{ \sum_{(m,w) \in \mu} p_m(w), \sum_{(m,w) \in \mu} p_w(m) \right\}. \]

Balanced Stable Marriage. Find a stable matching that attains \( \text{Bal} = \min_{\mu} bal(\mu) \).

Proposition. Balanced Stable Marriage is \( \text{NP-hard} \). [Feder, 1990]
Construction of a family of instances where no stable matching is both sex-equal and balanced. [Manlove, 2013]
Propositions. [Gupta, Saurabh and Zehavi, 2017]
1. Balanced Stable Marriage is W[1]-hard w.r.t \( tw \), the treewidth of the primal graph. Moreover, unless ETH fails, it cannot be solved in time \( f(tw)n^{o(tw)} \).
2. Balanced Stable Marriage can be solved in time \( n^{O(tw)} \).
Propositions. [Gupta, Saurabh and Zehavi, 2017]
1. Balanced Stable Marriage is solvable in time $2^{tw}n^{O(1)}$, where $tw$ is the treewidth of the rotation digraph.
2. Unless SETH fails, Balanced Stable Marriage cannot be solved in time $(2-\varepsilon)^{tw}n^{O(1)}$. 
\[ O_M = \sum_{(m,w) \in \mu_M} p_m(w); \quad O_W = \sum_{(m,w) \in \mu_W} p_w(m). \]

**Parameters.**

- \( t_1 = k - \min\{O_M, O_W\} \).
- \( t_2 = k - \max\{O_M, O_W\} \).
Propositions. [Gupta, Roy, Saurabh and Zehavi, 2017]

1. Balanced Stable Marriage admits a kernel with at most $3t_1$ men, $3t_1$ women, and such that each agent has at most $2t_1+1$ acceptable partners.

2. Balanced Stable Marriage is solvable in time $8^{t_1}n^{O(1)}$.

3. Balanced Stable Marriage is W[1]-hard w.r.t. $t_2$. 
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SMTI. Stable Marriage with Ties and Incomplete lists.

max-SMTI. Find a stable matching of max size.

min-SMTI. Find a stable matching of min size.
SMTI. Stable Marriage with Ties and Incomplete lists.

max-SMTI. Find a stable matching of max size.
min-SMTI. Find a stable matching of min size.

Proposition. max-SMTI and min-SMTI are NP-hard. [Irving, Iwama, Manlove, Miyazaki and Moitra, 2002]
Propositions. [Marx and Schlotter, 2010]
1. max-SMTI is FPT w.r.t. total length of ties.
2. max-SMTI is W[1]-hard w.r.t. number of ties.

In addition, they studied strict and permissive local search versions of max-SMTI.
Propositions. [Gupta, Saurabh and Zehavi, 2017]
1. max-SMTI and min-SMTI are \(W[1]\)-hard w.r.t \(tw\), the treewidth of the primal graph. Moreover, unless ETH fails, they cannot be solved in time \(f(tw)n^{o(tw)}\).
2. max-SMTI and min-SMTI can be solved in time \(n^{O(tw)}\).
Proposition. \textit{max-SMTI} and \textit{min-SMTI} admit a kernel of size $O(k^2)$, where $k$ is solution size.
[Adil, Gupta, Roy, Saurabh and Zehavi, 2017]
**Proposition.** $\text{max-SMTI}$ and $\text{min-SMTI}$ admit a kernel of size $O(k^2)$, where $k$ is solution size. [Adil, Gupta, Roy, Saurabh and Zehavi, 2017]

**Proposition.** $\text{max-SMTI}$ is FPT parameterized by number of "agent types". [Meeks and Rastegari, 2017]
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Stable Marriage with Covering Constraints (SMC). Given an instance of Stable Marriage with subsets $M^* \subseteq M$ and $W^* \subseteq W$, find a matching with minimum number of blocking pairs where $M^* \cup W^*$ are matched.
Mnich and Schlotter, 2017. Parameters:
- $b$ – # blocking pairs.
- $|M^*|, |W^*|$.  
- $\Delta_M (\Delta_W)$ – max length of lists of men (women).
Mnich and Schlotter, 2017. Parameters:

- $b$ – # blocking pairs.
- $|M^*|$, $|W^*|$.
- $\Delta_M (\Delta_W)$ – max length of lists of men (women).

Proposition. SMC is $W[1]$-hard w.r.t. $b+|W^*|$ even if $|M^*|=0$ and $\Delta_M = \Delta_W = 3$.

Proposition. SMC is FPT w.r.t. $b$ if $\Delta_W = 2$. Moreover, SMC is FPT w.r.t. $|M^*|+|W^*|$ if $\Delta_W = 2$. 
**Manipulation.** Stable Extension of Partial Matching (SEOPM).

**In.** Instance of Stable Marriage, partial matching $\mu$.

**Q.** Does there exist a set of lists for women, so that when this set is used, Gale-Shapley algorithm returns a matching that extends $\mu$?

In. Instance of Stable Marriage, partial matching $\mu$.

Q. Does there exist a set of lists for women, so that when this set is used, Gale-Shapley algorithm returns a matching that extends $\mu$?

Proposition. SEOPM is NP-hard. [Kobayashi and Matsui, 2010]

In. Instance of Stable Marriage, partial matching $\mu$.

Q. Does there exist a set of lists for women, so that when this set is used, Gale-Shapley algorithm returns a matching that extends $\mu$?

Proposition. SEOPM is solvable in time $2^{O(n)}$.
[Gupta and Roy, 2016]
max-Size min-BP SMI. Given an instance of Stable Marriage, find a maximum matching with minimum number of BPs among all maximum matchings.
max-Size min-BP SMI. Given an instance of Stable Marriage, find a maximum matching with minimum number of BPs among all maximum matchings.

Proposition. max-Size min-BP SMI is FPT parameterized by number of “agent types”. [Meeks and Rastegari, 2017]
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Stable Roommate. Given a set of agents, each ranking a subset of other agents, determine if there exists a stable matching.
Stable Roommate. Given a set of agents, each ranking a subset of other agents, determine if there exists a stable matching.

- Without ties, polynomial time. [Irving, 1985]
- With ties, NP-hard. [Ronn, 1990]
**Proposition.** Egalitarian Stable Roommate with Ties is FPT parameterized by $k-n$, where $k$ is the solution value. [Chen, Hermelin, Sorge and Yedidson, 2017] (Unmatched agents contribute, else para-NP-hard.)
Proposition. Egalitarian Stable Roommate with Ties is FPT parameterized by $k-n$, where $k$ is the solution value. [Chen, Hermelin, Sorge and Yedidson, 2017] (Unmatched agents contribute, else para-NP-hard.)

Proposition. min-BP Stable Roommate is W[1]-hard w.r.t. $b$, the number of blocking pairs. Moreover, unless ETH fails, it cannot be solved in time $f(b)n^{o(b)}$. [Chen, Hermelin, Sorge and Yedidson, 2017]
**Proposition.** \( \text{max-SRTI and min-SRTI admit a kernel of size } O(k^2), \) where \( k \) is the size of a maximum matching. [Adil, Gupta, Roy, Saurabh and Zehavi, ‘17]
**Proposition.** \( \text{max-SRTI} \) and \( \text{min-SRTI} \) admit a kernel of size \( O(k^2) \), where \( k \) is the size of a maximum matching. [Adil, Gupta, Roy, Saurabh and Zehavi, ‘17]

**Proposition.** \( \text{max-SRTI} \) is FPT w.r.t. number of ``agent types’’. [Meeks and Rastegari, 2017]
**Hospitals/Residents.** A set of hospitals $H$, and a set of residents $R$. Every agent has a ranked set of acceptable partners. Every hospital has a lower quota and an upper quota.

**Matching.** Every resident is matched at most once, and every hospital is matched according to its specification.
**Hospitals/Residents.** A set of hospitals $H$, and a set of residents $R$. Every agent has a ranked set of acceptable partners. Every hospital has a lower quota and an upper quota.

**Blocking pair** $(h,r)$. $h$ either has free space or prefers $r$ to an assigned resident; $r$ prefers being assigned to $h$ to the current status.
Hospitals/Residents. A set of hospitals $H$, and a set of residents $R$. Every agent has a ranked set of acceptable partners. Every hospital has a lower quota and an upper quota.

Goal. Does there exist a stable matching?
Hospitals/Residents. A set of hospitals $H$, and a set of residents $R$. Every agent has a ranked set of acceptable partners. Every hospital has a lower quota and an upper quota.

No ties: Polynomial-time.
Ties: NP-hard.
Proposition. Hospitals/Residents with Couples without lower quotas is W[1]-hard parameterized by the number of couples. [Marx and Schlotter, ‘11]

In addition, they studied strict and permissive local search versions of this problem.

Proposition. max-Hospitals/Residents is FPT w.r.t. number of ``agent types’’. [Meeks and Rastegari, 2017]
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Recent Advances in Parameterized Complexity will highlight recent exciting advances in the field of Parameterized Complexity, which is "growing by leaps and bounds". Moreover, to attract new researchers to work in this vibrant field, the event will also include a preparatory school. We thus invite both graduate students and established researchers to participate in Recent Advances in Parameterized Complexity. Limited travel support is available for graduate students and postdocs. A central theme in the program is future directions in Parameterized Complexity. The Parameterized Algorithms book will be freely distributed to registered participants.

Organizers:
- Fedor V. Fomin
- Daniel Lokshtanov
- Saket Saurabh
- Hadas Shachnai
- Meirav Zehavi

Invited Speakers:
- Piotr Faliszewski
- Daniel Marx
- Shmuel Onn
- Michal Pilipczuk
- Uri Zwick

More details?
rapctelaviv.weebly.com