Randomized Parameterized Algorithms

(and de-randomizations)

Color Coding

Design randomized algorithm first, then try to de-randomize it.

k-Path

Input: G, k

Question: Is there a path on k vertices in G?

Parameter: k

Will give an algorithm for k-path
with running time $(2e)^{k+o(k)}n^{O(1)}$.

Randomized Algorithm

Consider a random function $f : V(G) \rightarrow \{1...k\}$

For a set S on k vertices, what is the probability that all vertices get a different color?

Good colorings of S.

$$\frac{k!}{k^k} \geq {}^{1}/_{e^k}$$

Possible colorings of S.

Stirling approximation

Randomized Algorithm

Repeat $e^k \cdot t$ times:

Pick random $f : V(G) \rightarrow \{1...k\}$

Look for a colorful k-path.

For $x \geq 2$:

$$\frac{1}{4} \leq \left(1 - {}^{1}\!/_{x}\right)^{x} \leq \frac{1}{e}$$

If the algorithm finds a k-path, then G definitely has one.

If there is a k-path, the algorithm will find it with probability at least

$$1 - \left(1 - {}^{1}\!/_{e^k}\right)^{e^k \cdot t} \geq 1 - {}^{1}\!/_{e^t}$$

Finding a Colorful k-Path

«Exercise» Give a $k^{O(k)}$ time algorithm to determine whether a k-colored graph has a colorful k-path.

Finding a Colorful k-Path

Dynamic programming on the colors used by partial solutions.

vertex ↓

$$T[S,v] = \begin{cases} \textbf{true} & \text{If exists path on } |S| \text{ vertices ending in } v, \text{ using all colors from } S. \\ \textbf{false} & \text{otherwise} \end{cases}$$

↑ subset of {1...k}

Dynamic Programming

$$T[S,v] = \bigvee_{u \in N(v)} T[S \setminus f(v), u]$$

For each neighbor u of v

Is there a path ending in u that uses all colors in S, except v's color?

$2^k n$ table entries

O(n) time to fill each entry.

Total time: $2^k n^2$

Randomized Algorithm

Repeat $e^k \cdot 100$ times:

Pick random $f : V(G) \rightarrow \{1...k\}$

Look for a colorful k-path. ← Takes $2^k n^2$ time

If the algorithm finds a k-path, then G definitely has one.

If there is a k-path, the algorithm will find it with probability at least $1 - {}^{1}/_{e^{100}}$

Total time: $O((2e)^k n^2)$.

De-randomization

How can we make the algorithm deterministic?

Let $F = f_1 \ldots f_t$ be a family of functions with
$f_i: V(G) \rightarrow \{1 \ldots k\}$.

F is a k-universal hash family F if for every set
$S \subseteq V(G)$ of size k, there is an $f_i \in F$ such that f_i makes S colorful.

Deterministic Algorithm

Construct a k-universal hash family F. ← Takes t time

For each $f \in F$:

Look for a colorful k-path. ← Takes $2^k n^2$ time

If the algorithm finds a k-path, then G definitely has one.

If there is a k-path, the algorithm will find it.

Total time: $O(t + |F| \cdot 2^k n^2)$.

Constructing Hash Functions

[NSS'95] Can construct a k-perfect hash family F of size $e^{k+o(k)} \log n$ in time $e^{k+o(k)} n \log n$.

k-Path in time $(2e)^{k+o(k)} n^{O(1)} \leq 5.44^k n^{O(1)}$

Random Separation: Set Splitting

Input: Family $S_1 \ldots S_m$ of sets over a universe $U = v_1 \ldots v_n$, integer k.

Question: Is there a coloring $c : U \rightarrow \{0,1\}$ such that at least k sets contain an element colored 0 and an element colored 1?

Parameter: k

Will give a $2^k n^{O(1)}$ time randomized, and a $4^k n^{O(1)}$ time deterministic algorithm.

Randomized Algorithm

Pick a random coloring $c : V(G) \rightarrow \{1,2\}$.

If c splits at least k sets, return c.

If the algorithm returns a coloring, then it is correct.

Claim: If there is a coloring ψ that splits at least k sets, then a random coloring will split at least k sets with probability at least ${}^{1}/_{2^k}$

The $2^k n^{O(1)}$ time randomized algorithm follows directly from the claim.

Proof of claim

Suppose ψ splits the sets $S_1 \ldots S_k$.

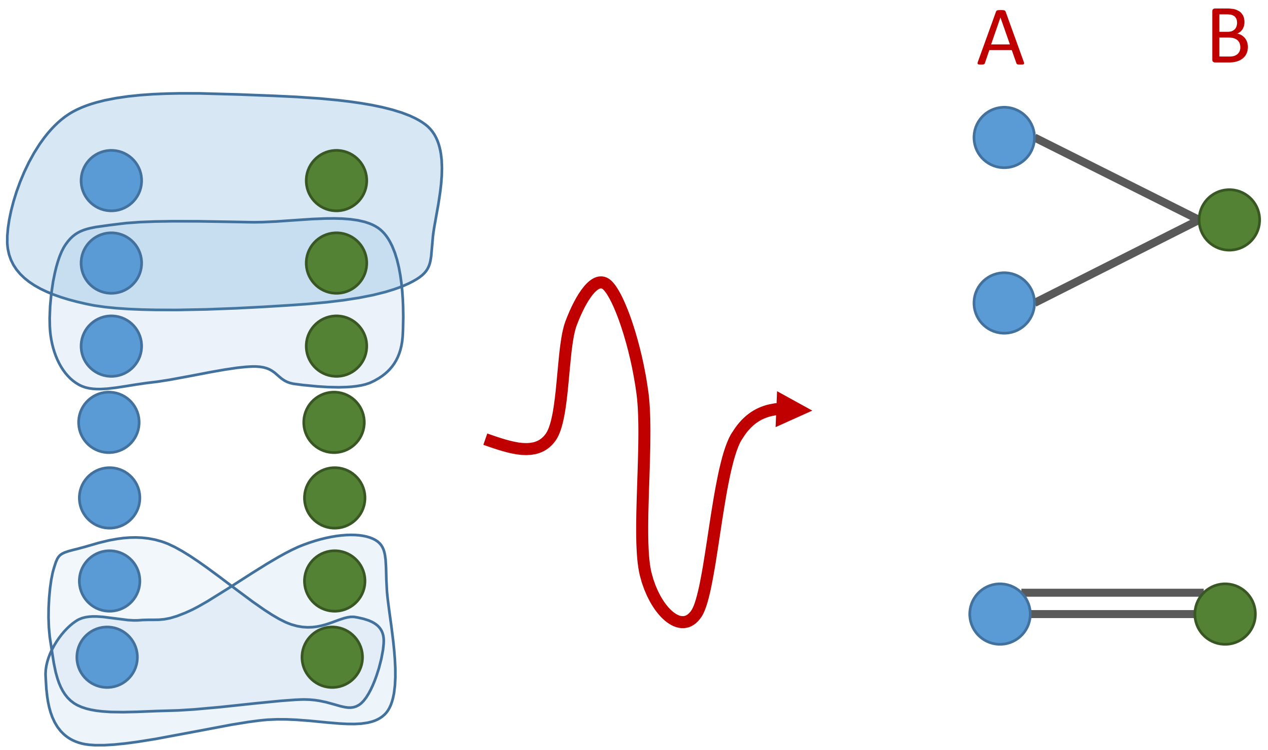
Make a bipartite graph $G=(A \cup B, E)$ as follows:

- A is a minimal hitting set for $S_1 \ldots S_k$ colored 0
- B is a minimal hitting set for $S_1 \ldots S_k$ colored 1

For i from 1 to k

Add one edge between a vertex in $S_i \cap A$ and a vertex in $S_i \cap B$.

Set Splitting Graph



Proof of claim, continued.

If c properly colors G then all sets $S_1 \ldots S_k$ are split.

G has $\leq k$ edges and at most $\leq 2k$ vertices

G has $k+x$ vertices → G has $\geq x$ components

Probability that c properly colors G is at least:

Number of proper colorings → numerator

Number of colorings → denominator

$$\frac{2^x}{2^{k+x}} = \frac{1}{2^k}$$

Set Splitting Randomized Algorithm

Repeat $100 \cdot 2^k$ times:

Pick a random coloring $c : V(G) \rightarrow \{1...k\}$.

If c splits at least k sets, return c.

Running time: $O(2^k nm)$

If the algorithm returns a coloring, then it is correct.

If there is a coloring that splits k sets, the algorithm will find one with probability $1-1/2^{100}$.

Universal Coloring Family

Let $F = \{c_1 \ldots c_t\}$ be a family of colorings $V(G) \rightarrow \{0,1\}$

F is a k-universal coloring family if for every set S on at most k vertices and every way of coloring S there is some $c_i \in F$ which colors S exactly like that.

Set Splitting Algorithm

Construct 2k-universal coloring family F ← Takes t time

For each $c \in F$**:**

If c splits at least k sets, **return** c.

If the algorithm returns a coloring, then it is correct.

If there is a coloring that splits k sets, the algorithm will find one, since the graph G has $\leq 2k$ vertices.

Running time: $O(t + |F|nm)$

Construction

of Universal Coloring Families

[NSS'95] Can construct a k-universal coloring family F of size $2^{k+o(k)}\log n$ in time $2^{k+o(k)} n \log n$.

(We need a 2k-universal coloring family)

Set Splitting in time $4^{k+o(k)} n^{O(1)}$.

Induced Subgraph Isomorphism

Input: Graphs G and H, G has maximum degree Δ, |V(H)| = k

Question: Does G contain H as an induced subgraph?

Parameters: Δ + |V(H)|

Encodes the k-Clique problem

(so FPT just by k is unlikely)

Naive algorithm: $n^{O(k)}$

|V(G)|

Will see a $\Delta^{O(k)}$ time algorithm

Random Separation

Will assume H is **connected**

Color vertices of G red with probability p,
blue with probability 1-p

Delete all blue vertices

Determine whether any (red) connected component
is equal to H using
Graph Isomorphism in time $2^{\text{polylog}(k)}$

Success Probability

If G does not contain H then algorithm always says no

If G contains H then:

All the vertices of H are colored red with probability p^k

All the neighbors of H (in G) are colored blue
with probability at least $(1-p)^{\Delta k}$

Success probability: $p^k(1-p)^{\Delta k} = \frac{1}{\Delta^k}(1-\frac{1}{\Delta})^{\Delta k} \geq \frac{1}{(4\Delta)^k}$

Set $p = 1/\Delta$

Running time

Each run of the algorithm takes $2^{o(k)}$ time.

Repeat $(4\Delta)^k$ times for constant success probability.

Total runtime: $(4\Delta)^{k+o(k)}$

Derandomization

Universal Coloring Families → deterministic algorithm for Induced Subgraph Isomorphism with running time $2^{O(\Delta k)}$.

This can be improved to $\Delta^{O(k)}$.

Other variants

Simple extension 1: Algorithm for the case where H is not necessarily connected with essentially the same running time.

Simple extension 2: Algorithm for Subgraph Isomorphism problem (not induced) with similar ish running time.

Feedback Vertex Set

Feedback Vertex Set (FVS)

IN: G, k

Q: Is there a set S of ≤ k vertices such that G\S is a forest?

FVS reduction rules

R1: Delete vertices of degee ≤ 1

R2: Replace degree 2 vertices by edges (keep multiedges)

α-cover Lemma

A set $S \subseteq V(G)$ is an α-cover if

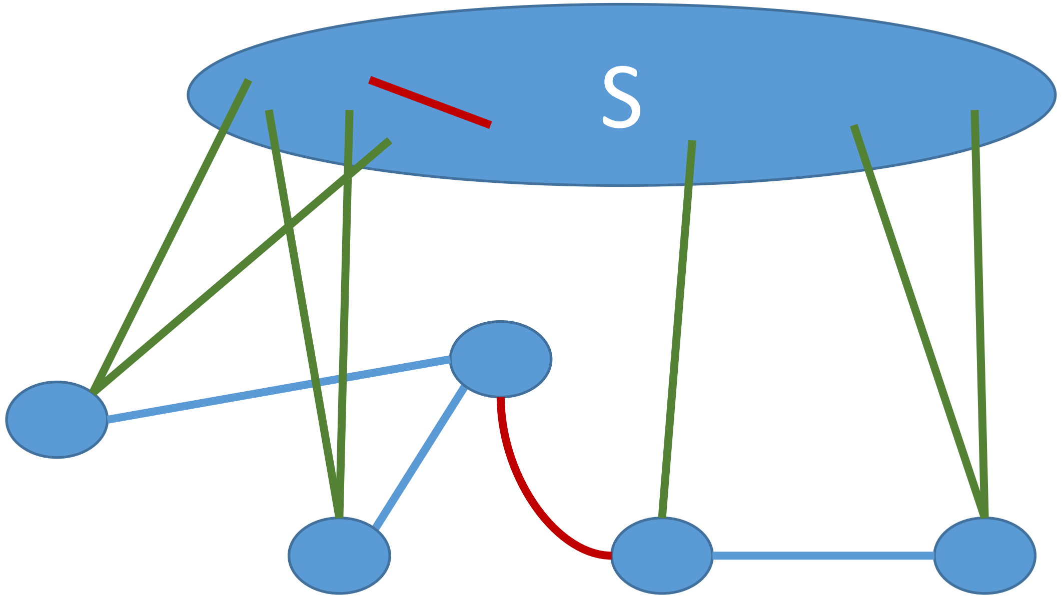
$$\Sigma_{v \in S} d(v) \geq \alpha \cdot \Sigma_{v \in V(G)} d(v) \ (= \alpha \cdot 2m)$$

Lemma: If R1 and R2 do not apply, then every feedback vertex set S of G is a ¼-cover.

α-cover Lemma

Lemma: If R1 and R2 do not apply then every feedback vertex is a ¼-cover.

Proof:



$$\frac{\Sigma_{v \in S} d(v) \quad -2}{\Sigma_{v \in V(G)} d(v) \quad +2}$$

Algorithm for FVS

while G is not empty

Apply R1 and R2 on G exhaustively

Select a vertex v with prob = d(v) / 2m.

S := S U {v}

G := G\v

If |S| > k **output** NO

output S

Succeeds with probability ¼

Feedback Vertex Set

Runs in $O(k(n+m))$ time, succeeds with $\frac{1}{4^k}$ probability.

So $O(4^k k(n+m))$ time for constant success probability.

Expected output solution size is $\leq$ 4OPT, this is a 4-approximation!

Feedback Vertex Set below 2^n

Feedback Vertex Set (FVS)

IN: G, k

Q: Is there a set S of ≤ k vertices such that G\S is a forest?

Saw a $4^k n^{O(1)}$ time algorithm

Can we beat 2^n?

Branching is tricky

Next: $O((2-\frac{1}{4})^n) = O(1.75^n)$ time using $4^k n^{O(1)}$ as black box

Feedback Vertex Set algorithm

Given n, k, pick integer $t \leq k$.

Pick a set S of size t uniformly at random, put S in solution.

(By deleting S and decreasing parameter by t)

Try to extend S to a feedback vertex set of size $\leq k$

(By running $4^k n^{O(1)}$ time algorithm on $(G-S, k-t)$)

Analysis

Success probability: $\frac{\binom{k}{t}}{\binom{n}{t}}$

Number of subsets of solution of size t

Number of subsets of size t

Running time: $4^{k-t}n^{O(1)}$

Running time for constant success probability: $\frac{\binom{n}{t}}{\binom{k}{t}}4^{k-t}$

Given n, consider the worst k:

$$\max_{k \leq n} \min_{t \leq k} \frac{\binom{n}{t}}{\binom{k}{t}}4^{k-t}$$

Given n and k pick t so that running time is minimized:

$$\min_{t \leq k} \frac{\binom{n}{t}}{\binom{k}{t}}4^{k-t}$$

Analysis, cont'd

Lemma: For all $c > 1$, $\max_{k \le n} \min_{t \le k} \frac{\binom{n}{t}}{\binom{k}{t}} c^{k-t} \le O((2 - \frac{1}{c})^n)$

Corollary: Feedback Vertex Set in $O((2 - \frac{1}{4})^n)$ = $O(1.75^n)$ time.

Generalizing

Nothing in the algorithm / analysis was specific to FVS!

Look for a set of size k in a universe of size n.

If we can extend a set of size t to a solution of size k in time $c^{k-t}n^{O(1)}$ then

We can find a solution in time $O((2-\frac{1}{c})^n)$

Can be fully derandomized.

Chromatic Coding

Random Separation

Techiques

Picking Random Solution Vertices

Color Coding

Mod 2 Counting + Isolation


Thank you!